Stabilizing a Flexible Beam on a Cart: A Distributed Port Hamiltonian Approach

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Outline

Infinite Dimensional Port Hamiltonian System

- $\mathcal{D}$: n-dimensional spatial domain
- $\partial\mathcal{D}$: $(n-1)$-dimensional boundary of $\mathcal{D}$
- *Space of Flows*:

$$\mathcal{F}_{p,q} := \Omega^p(\mathcal{D}) \times \Omega^q(\mathcal{D}) \times \Omega^{n-p}(\partial\mathcal{D})$$

- *Space of Efforts*:

$$\mathcal{E}_{p,q} := \Omega^{n-p}(\mathcal{D}) \times \Omega^{n-q}(\mathcal{D}) \times \Omega^{n-q}(\partial\mathcal{D})$$

Dirac Structure

$$\mathbb{D} = \left\{ (f_p, f_q, f_b, e_p, e_q, e_b) \in \mathcal{F}_{p,q} \times \mathcal{E}_{p,q} | \begin{bmatrix} f_p \\ f_q \end{bmatrix} = \mathbb{J} \begin{bmatrix} e_p \\ e_q \end{bmatrix}, \begin{bmatrix} f_b \\ e_b \end{bmatrix} = \mathbb{G} \begin{bmatrix} e_p|_{\partial\mathcal{D}} \\ e_q|_{\partial\mathcal{D}} \end{bmatrix} \right\} \quad (1)$$

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$$\mathbb{D} = \left\{ (f_p, f_q, f_b, e_p, e_q, e_b) \in \mathcal{F}_{p,q} \times \mathcal{E}_{p,q} \middle| \begin{bmatrix} f_p \\ f_q \end{bmatrix} = \mathbb{J} \begin{bmatrix} e_p \\ e_q \end{bmatrix}, \begin{bmatrix} f_b \\ e_b \end{bmatrix} = \mathbb{G} \begin{bmatrix} e_p|_{\partial\mathcal{D}} \\ e_q|_{\partial\mathcal{D}} \end{bmatrix} \right\} \quad (1)$$

Distributed Port Hamiltonian System

- State-space : $\Omega^p(\mathcal{D}) \times \Omega^q(\mathcal{D})$
- Hamiltonian : $\mathcal{H}$
- $$\begin{bmatrix} \frac{\partial \alpha_p}{\partial t} \\ \frac{\partial \alpha_q}{\partial t} \end{bmatrix} = -\mathbb{J} \begin{bmatrix} \delta_{\alpha_p}\mathcal{H} \\ \delta_{\alpha_q}\mathcal{H} \end{bmatrix}; \quad \begin{bmatrix} f_b \\ e_b \end{bmatrix} = \mathbb{G} \begin{bmatrix} \delta_{\alpha_p}\mathcal{H}|_{\partial\mathcal{D}} \\ \delta_{\alpha_q}\mathcal{H}|_{\partial\mathcal{D}} \end{bmatrix} \quad (2)$$

System Description

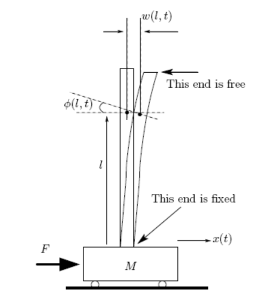

FIGURE: A flexible beam on a cart

- Spatial domain : $\mathcal{D} := [0, L]$
- Boundary of the domain : $\partial\mathcal{D} = \{0, L\}$
- $\phi(0,t) = 0 \ \forall t \geq 0$
- $w(0,t) = 0 \ \forall t \geq 0$

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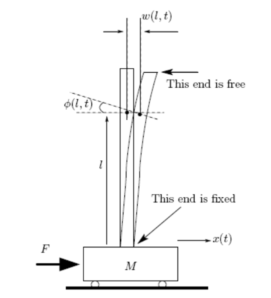

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Hamiltonian

$$\mathcal{H} = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}\int_0^L \left[\rho\left(\frac{\partial w}{\partial t} + \dot{x}\right)^2 + I_\rho\left(\frac{\partial \phi}{\partial t}\right)^2 + K\left(\phi - \frac{\partial w}{\partial l}\right)^2 + EI\left(\frac{\partial \phi}{\partial l}\right)^2\right] dl + \frac{\rho L^2 g}{2}$$

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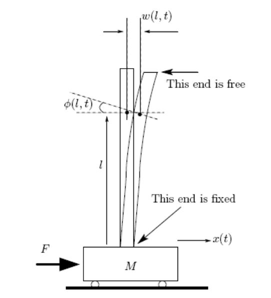

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Assumption

We assume: $z \triangleq w + x$

Then

$$\begin{aligned} \frac{\partial z}{\partial t} &= \frac{\partial w}{\partial t} + \dot{x} \\ \frac{\partial z}{\partial l} &= \frac{\partial w}{\partial l} \\ z(0,t) &= x(t) \qquad \forall t \end{aligned}$$

Port Hamiltonian Model for The Flexible Beam 1

1-forms on $\mathcal{D}$

$$
\begin{aligned}
\epsilon_t(l,t) &\triangleq \left(\frac{\partial z}{\partial l} - \phi\right) dl \\
\epsilon_r(l,t) &\triangleq \left(\frac{\partial \phi}{\partial l}\right) dl \\
p_t(l,t) &\triangleq \rho \left(\frac{\partial z}{\partial t}\right) dl \\
p_r(l,t) &\triangleq I_\rho \left(\frac{\partial \phi}{\partial t}\right) dl
\end{aligned}
$$

Port Hamiltonian Model for The Flexible Beam 1

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p_t(l,t) &\triangleq \rho\left(\frac{\partial z}{\partial t}\right) dl \\
p_r(l,t) &\triangleq I_\rho\left(\frac{\partial \phi}{\partial t}\right) dl
\end{aligned}$$

Beam Hamiltonian

$$\mathcal{H}_\mathcal{B} = \int_\mathcal{D} \left[\frac{1}{\rho}(\star p_t) \wedge p_t + \frac{1}{I_\rho}(\star p_r) \wedge p_r + K(\star\epsilon_t) \wedge \epsilon_t + EI(\star\epsilon_r) \wedge \epsilon_r\right] + \frac{1}{2}\rho g L^2$$

$$\frac{d\mathcal{H}_\mathcal{B}}{dt} = \int_\mathcal{D} \left[(\frac{1}{\rho} \star p_t) \wedge \frac{\partial p_t}{\partial t} + (\frac{1}{I_\rho} \star p_r) \wedge \frac{\partial p_r}{\partial t} + (K \star \epsilon_t) \wedge \frac{\partial \epsilon_t}{\partial t} + (EI \star \epsilon_r) \wedge \frac{\partial \epsilon_r}{\partial t}\right]$$

Port Hamiltonian Model for The Flexible Beam 2

Dirac Structure

- **Flows**:
  $\mathcal{F} := \Psi^1(\mathcal{D}) \times \Psi^1(\mathcal{D}) \times \Psi^1(\mathcal{D}) \times \Psi^1(\mathcal{D}) \times (\Psi^0(\partial\mathcal{D}) \times \Psi^0(\partial\mathcal{D})) \times (\Psi^0(\partial\mathcal{D}) \times \Psi^0(\partial\mathcal{D}))$
- **Efforts**:
  $\mathcal{E} := \Psi^0(\mathcal{D}) \times \Psi^0(\mathcal{D}) \times \Psi^0(\mathcal{D}) \times \Psi^0(\mathcal{D}) \times (\Psi^0(\partial\mathcal{D}) \times \Psi^0(\partial\mathcal{D})) \times (\Psi^0(\partial\mathcal{D}) \times \Psi^0(\partial\mathcal{D}))$
- $\mathbb{D} \subset \mathcal{F} \times \mathcal{E}$, $\mathbb{D} = \mathbb{D}^{\perp}$

$$\begin{aligned} \mathbb{D} \quad = \quad & \Bigg\{ (f_{p_t}, f_{p_r}, f_{\epsilon_t}, f_{\epsilon_r}, f_b^t, f_b^r, e_{p_t}, e_{p_r}, e_{\epsilon_t}, e_{\epsilon_r}, e_b^t, e_b^r) \in \mathcal{F} \times \mathcal{E} | \\ & \begin{bmatrix} f_{p_t} \\ f_{p_r} \\ f_{\epsilon_t} \\ f_{\epsilon_r} \end{bmatrix} = -\begin{bmatrix} 0 & 0 & d & 0 \\ 0 & 0 & \star & d \\ d & -\star & 0 & 0 \\ 0 & d & 0 & 0 \end{bmatrix} \begin{bmatrix} e_{p_t} \\ e_{p_r} \\ e_{\epsilon_t} \\ e_{\epsilon_r} \end{bmatrix}; \begin{bmatrix} f_b^t \\ f_b^r \\ e_b^t \\ e_b^r \end{bmatrix} = \begin{bmatrix} e_{p_t}|_{\partial\mathcal{D}} \\ e_{p_r}|_{\partial\mathcal{D}} \\ e_{\epsilon_t}|_{\partial\mathcal{D}} \\ e_{\epsilon_r}|_{\partial\mathcal{D}} \end{bmatrix} \Bigg\} \end{aligned} \quad (3)$$

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$\mathcal{F} := \Psi^1(\mathcal{D}) \times \Psi^1(\mathcal{D}) \times \Psi^1(\mathcal{D}) \times \Psi^1(\mathcal{D}) \times (\Psi^0(\partial\mathcal{D}) \times \Psi^0(\partial\mathcal{D})) \times (\Psi^0(\partial\mathcal{D}) \times \Psi^0(\partial\mathcal{D}))$
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$$\mathbb{D} = \left\{ (f_{p_t}, f_{p_r}, f_{\epsilon_t}, f_{\epsilon_r}, f_b^t, f_b^r, e_{p_t}, e_{p_r}, e_{\epsilon_t}, e_{\epsilon_r}, e_b^t, e_b^r) \in \mathcal{F} \times \mathcal{E} \right|$$

$$\left. \begin{bmatrix} f_{p_t} \\ f_{p_r} \\ f_{\epsilon_t} \\ f_{\epsilon_r} \end{bmatrix} = -\begin{bmatrix} 0 & 0 & d & 0 \\ 0 & 0 & \star & d \\ d & -\star & 0 & 0 \\ 0 & d & 0 & 0 \end{bmatrix} \begin{bmatrix} e_{p_t} \\ e_{p_r} \\ e_{\epsilon_t} \\ e_{\epsilon_r} \end{bmatrix}; \begin{bmatrix} f_b^t \\ f_b^r \\ e_b^t \\ e_b^r \end{bmatrix} = \begin{bmatrix} e_{p_t}|_{\partial\mathcal{D}} \\ e_{p_r}|_{\partial\mathcal{D}} \\ e_{\epsilon_t}|_{\partial\mathcal{D}} \\ e_{\epsilon_r}|_{\partial\mathcal{D}} \end{bmatrix} \right\} \quad (3)$$

Distributed Port Hamiltonian Model

$$\frac{\partial}{\partial t} \begin{bmatrix} p_t \\ p_r \\ \epsilon_t \\ \epsilon_r \end{bmatrix} = \begin{bmatrix} 0 & 0 & d & 0 \\ 0 & 0 & \star & d \\ d & -\star & 0 & 0 \\ 0 & d & 0 & 0 \end{bmatrix} \begin{bmatrix} \delta_{p_t}\mathcal{H}_\mathcal{B} \\ \delta_{p_r}\mathcal{H}_\mathcal{B} \\ \delta_{\epsilon_t}\mathcal{H}_\mathcal{B} \\ \delta_{\epsilon_r}\mathcal{H}_\mathcal{B} \end{bmatrix}; \begin{bmatrix} f_b^t \\ f_b^r \\ e_b^t \\ e_b^r \end{bmatrix} = \begin{bmatrix} \delta_{p_t}\mathcal{H}_\mathcal{B}|_{\partial\mathcal{D}} \\ \delta_{p_r}\mathcal{H}_\mathcal{B}|_{\partial\mathcal{D}} \\ \delta_{\epsilon_t}\mathcal{H}_\mathcal{B}|_{\partial\mathcal{D}} \\ \delta_{\epsilon_r}\mathcal{H}_\mathcal{B}|_{\partial\mathcal{D}} \end{bmatrix} \quad (4)$$

The Cart and The Controller Model

- $H_c = H_{cart} + H_{controller}$

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Port Hamiltonian Model of Combined System

$$\begin{bmatrix} \dot{q}_c \\ \dot{p}_c \end{bmatrix} = \left(\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & D_c \end{bmatrix} \right) \begin{bmatrix} \frac{\partial H_c}{\partial q_c} \\ \frac{\partial H_c}{\partial p_c} \end{bmatrix} + \begin{bmatrix} 0 \\ G_c \end{bmatrix} f_c$$

$$\text{and,} \quad e_c = G_c^T \partial_{p_c} H_c \tag{5}$$

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- $q_c = [q_{c_1}, q_{c_2}]^T \in Q_C \subset \mathbb{R}^2$
- $D_c = D_c^T > 0$

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- $D_c = D_c^T > 0$

Motivation

Separation of the finite and infinite dimensional parts of the overall system

Power Conserving Interconnection

$$f_c^T e_c = f_b(0) \wedge e_b(0) - f_b(L) \wedge e_b(L) \quad (6)$$

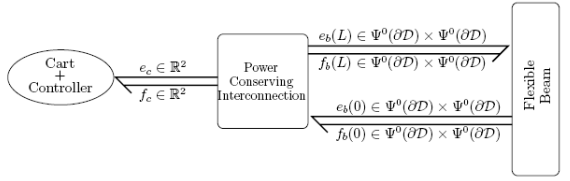

Figure: Bond-graph representation of the closed-loop system

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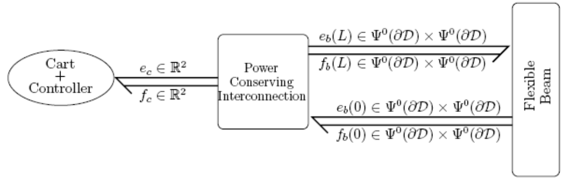

FIGURE: Bond-graph representation of the closed-loop system

- **Closed Loop Hamiltonian**:

$$\mathcal{H}_{cl} := \mathcal{H}_{\mathcal{B}} + H_c \quad (7)$$

- **Extended Configuration Space**:

$$\mathcal{X}_{cl} := \underbrace{Q_c \times T^* Q_c}_{\mathcal{X}_c} \times \underbrace{\Psi^1(\mathcal{D}) \times \Psi^1(\mathcal{D}) \times \Psi^1(\mathcal{D}) \times \Psi^1(\mathcal{D})}_{\mathcal{X}_\infty} \quad (8)$$

Energy Casimir Method

Motivation

- We aren't interested about the cart's position or the controller's configuration.
- Hence, the equilibrium is defined in the $\mathcal{X}_{cl}/Q_C$ space.
- The energy-Casimir approach is adopted to stabilize the relative equilibria [2].

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Definition

(Casimir functionals) *Consider a scalar function $\mathcal{C} : \mathcal{X}_{cl} \rightarrow \mathbb{R}$ defined on the extended configuration space (8). Then $\mathcal{C}$ is a Casimir functional for the closed loop system if and only if*

$$\frac{d\mathcal{C}}{dt} = 0 \quad \forall \mathcal{H}_{cl} : \mathcal{X}_{cl} \rightarrow \mathbb{R} \tag{9}$$

where $\mathcal{H}_{cl} = \mathcal{H}_B + H_c$. (Recall $H_c = H_{cart} + H_{controller}$)

Sufficient Conditions for A Casimir

We Have,

$$\frac{d\mathcal{C}}{dt} = \left(\frac{\partial \mathcal{C}}{\partial q_c}\right)^T \dot{q_c} + \left(\frac{\partial \mathcal{C}}{\partial p_c}\right)^T \dot{p_c} + \int_{\mathcal{D}} \left[\delta_{p_t}\mathcal{C} \wedge \frac{\partial p_t}{\partial t} + \delta_{p_r}\mathcal{C} \wedge \frac{\partial p_r}{\partial t} + \delta_{\epsilon_t}\mathcal{C} \wedge \frac{\partial \epsilon_t}{\partial t} + \delta_{\epsilon_r}\mathcal{C} \wedge \frac{\partial \epsilon_r}{\partial t}\right]$$

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Sufficient Conditions for Existence of Casimir

1. $d(\delta_{p_t}\mathcal{C}) - *\delta_{p_r}\mathcal{C} = 0$
2. $d(\delta_{p_r}\mathcal{C}) = 0$
3. $d(\delta_{\epsilon_t}\mathcal{C}) = 0$
4. $d(\delta_{\epsilon_r}\mathcal{C}) + *\delta_{\epsilon_t}\mathcal{C} = 0$
5. $\frac{\partial \mathcal{C}}{\partial p_c} = 0$
6. $\left(\frac{\partial \mathcal{C}}{\partial q_c}\right)^T \frac{\partial H_c}{\partial p_c} + \delta_{p_t}\mathcal{C}\big|_L \wedge \delta_{\epsilon_t}\mathcal{H}_{\mathcal{B}}\big|_L - \delta_{p_t}\mathcal{C}\big|_0 \wedge \delta_{\epsilon_t}\mathcal{H}_{\mathcal{B}}\big|_0 + \delta_{p_r}\mathcal{C}\big|_L \wedge \delta_{\epsilon_r}\mathcal{H}_{\mathcal{B}}\big|_L - \delta_{p_r}\mathcal{C}\big|_0 \wedge \delta_{\epsilon_r}\mathcal{H}_{\mathcal{B}}\big|_0 + \delta_{\epsilon_t}\mathcal{C}\big|_L \wedge \delta_{p_t}\mathcal{H}_{\mathcal{B}}\big|_L - \delta_{\epsilon_t}\mathcal{C}\big|_0 \wedge \delta_{p_t}\mathcal{H}_{\mathcal{B}}\big|_0 + \delta_{\epsilon_r}\mathcal{C}\big|_L \wedge \delta_{p_r}\mathcal{H}_{\mathcal{B}}\big|_L - \delta_{\epsilon_r}\mathcal{C}\big|_0 \wedge \delta_{p_r}\mathcal{H}_{\mathcal{B}}\big|_0 = 0$

Casimir Functional for The System of Our Interest 1

Assumed Form of Casimir

$$\mathcal{C}_i(q_c, p_c, p_t, p_r, \epsilon_t, \epsilon_r) := q_{c_i} + \tilde{\mathcal{C}}_i(p_c, p_t, p_r, \epsilon_t, \epsilon_r) \quad i = 1, 2$$

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Casimir Functionals

1. $\mathcal{C}_1 = q_{c_1} + \int_{\mathcal{D}} \left[(k_3^1 + k_1^1 l)p_t + k_1^1 p_r + k_2^1 \epsilon_t + (k_4^1 - k_2^1 l)\epsilon_r\right]$
2. $\mathcal{C}_2 = q_{c_2} + \int_{\mathcal{D}} \left[(k_3^2 + k_1^2 l)p_t + k_1^2 p_r + k_2^2 \epsilon_t + (k_4^2 - k_2^2 l)\epsilon_r\right]$

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2. $\mathcal{C}_2 = q_{c_2} + \int_{\mathcal{D}} \left[(k_3^2 + k_1^2 l) p_t + k_1^2 p_r + k_2^2 \epsilon_t + (k_4^2 - k_2^2 l) \epsilon_r \right]$

where,

- $k_1^i = \delta_{p_r} \mathcal{C}_i \big|_{\partial\mathcal{D}=0}$
- $k_2^i = \delta_{\epsilon_t} \mathcal{C}_i \big|_{\partial\mathcal{D}=0}$
- $k_3^i = \delta_{p_t} \mathcal{C}_i \big|_{\partial\mathcal{D}=0}$
- $k_4^i = \delta_{\epsilon_r} \mathcal{C}_i \big|_{\partial\mathcal{D}=0}$

for $i = 1, 2$

Casimir Functional for The System of Our Interest 2

Controller Effort

$$
\begin{aligned}
e_{c_i} \quad = \quad & \begin{bmatrix} k_3^1 & k_1^1 \\ k_3^2 & k_1^2 \end{bmatrix} \begin{bmatrix} \delta_{\epsilon_t}\mathcal{H}_{\mathcal{B}}|_{\partial\mathcal{D}=0} \\ \delta_{\epsilon_r}\mathcal{H}_{\mathcal{B}}|_{\partial\mathcal{D}=0} \end{bmatrix} + \begin{bmatrix} k_2^1 & k_4^1 \\ k_2^1 & k_4^1 \end{bmatrix} \begin{bmatrix} \dot{x} \\ 0 \end{bmatrix} \\
& - \begin{bmatrix} k_2^1 & k_4^1 - k_2^1 L \\ k_2^2 & k_4^2 - k_2^2 L \end{bmatrix} \begin{bmatrix} \delta_{p_t}\mathcal{H}_{\mathcal{B}}|_{\partial\mathcal{D}=L} \\ \delta_{p_r}\mathcal{H}_{\mathcal{B}}|_{\partial\mathcal{D}=L} \end{bmatrix}
\end{aligned} \tag{10}
$$

Casimir Functional for The System of Our Interest 2

Controller Effort

$$
\begin{aligned}
e_{c_i} = & \begin{bmatrix} k_3^1 & k_1^1 \\ k_3^2 & k_1^2 \end{bmatrix} \begin{bmatrix} \delta_{\epsilon_t}\mathcal{H}_{\mathcal{B}}|_{\partial\mathcal{D}=0} \\ \delta_{\epsilon_r}\mathcal{H}_{\mathcal{B}}|_{\partial\mathcal{D}=0} \end{bmatrix} + \begin{bmatrix} k_2^1 & k_4^1 \\ k_2^1 & k_4^1 \end{bmatrix} \begin{bmatrix} \dot{x} \\ 0 \end{bmatrix} \\
& - \begin{bmatrix} k_2^1 & k_4^1 - k_2^1 L \\ k_2^2 & k_4^2 - k_2^2 L \end{bmatrix} \begin{bmatrix} \delta_{p_t}\mathcal{H}_{\mathcal{B}}|_{\partial\mathcal{D}=L} \\ \delta_{p_r}\mathcal{H}_{\mathcal{B}}|_{\partial\mathcal{D}=L} \end{bmatrix} \qquad (10)
\end{aligned}
$$

Also,

$$
G_c = \left(\frac{\partial \mathcal{C}}{\partial q_c}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad (11)
$$

Equilibrium Configuration

Equilibrium Configuration of The Closed Loop System

• Desired equilibrium configuration of the flexible beam is the vertically upright position.

- $p_t^* = 0$
- $p_r^* = 0$
- $\epsilon_t^* = 0$
- $\epsilon_r^* = 0$
- $q_{c_1}^* = \mathcal{C}_1(\mathcal{X}^0)$
- $q_{c_2}^* = \mathcal{C}_2(\mathcal{X}^0)$
- $p_c^* = 0$

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- $q_{c_1}^* = \mathcal{C}_1(\mathcal{X}^0)$
- $q_{c_2}^* = \mathcal{C}_2(\mathcal{X}^0)$
- $p_c^* = 0$

Controller Hamiltonian (H_c)

$$H_c = \frac{1}{2} {p_c}^T {M_c}^{-1} p_c + \frac{1}{2} K_{c1} (q_{c_1} - q_{c_1}^*)^2 + \frac{1}{2} K_{c2} (q_{c_2} - q_{c_2}^*)^2 \quad (12)$$

- $M_c = M_c^T > 0$
- $K_{c1} > 0$, $K_{c2} > 0$

Stability Analysis 1

Definition

(Lyapunov Stability for Mixed Finite and Infinite Dimensional Systems) *[3] The equilibrium configuration $\mathcal{X}^*$ for a mixed finite and infinite dimensional system is said to be stable in the sense of Lyapunov with respect to the norm $||\cdot||$ if, for every $\epsilon > 0$, there exists $\delta_\epsilon > 0$ such that*

$$||\mathcal{X}(0) - \mathcal{X}^*|| < \delta_\epsilon \Rightarrow ||\mathcal{X}(t) - \mathcal{X}^*|| < \epsilon$$

for every $t > 0$, where $\mathcal{X}(0)$ is the initial configuration of the mixed system.

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Conditions for Stable Equilibrium

1. The equilibrium configuration ($\mathcal{X}^*$) should be an extremum of the closed-loop Hamiltonian ($\mathcal{H}_{cl}$); that is,

$$\nabla \mathcal{H}_{cl}(\mathcal{X}^*) = 0 \tag{13}$$

2. **Convexity condition**: There exists some $\gamma_1, \gamma_2, \alpha > 0$ such that

$$\gamma_1 \|\Delta \mathcal{X}\|^2 \leq \mathcal{N}(\Delta \mathcal{X}) \leq \gamma_2 \|\Delta \mathcal{X}\|^\alpha \tag{14}$$

   where

$$\mathcal{N}(\Delta \mathcal{X}) = \mathcal{H}_{cl}(\mathcal{X}^* + \Delta \mathcal{X}) - \mathcal{H}_{cl}(\mathcal{X}^*)$$

Stability Analysis 2

Definition for The *Norm*

$$\|\Delta\mathcal{X}\|^2 = \int_{\mathcal{D}} [(\star\Delta p_t)\wedge\Delta p_t + (\star\Delta p_r)\wedge\Delta p_r + (\star\Delta\epsilon_t)\wedge\ \Delta\epsilon_t + (\star\Delta\epsilon_r)\wedge\Delta\epsilon_r] + \Delta {p_c}^T\Delta p_c \qquad (15)$$

Stability Analysis

Definition for The *Norm*

$$\|\Delta\mathcal{X}\|^2 = \int_{\mathcal{D}} \left[(\star\Delta p_t)\wedge\Delta p_t + (\star\Delta p_r)\wedge\Delta p_r + (\star\Delta\epsilon_t)\wedge\ \Delta\epsilon_t + (\star\Delta\epsilon_r)\wedge\Delta\epsilon_r\right] + \Delta p_c{}^T\Delta p_c \tag{15}$$

Expression for $\mathcal{N}(\Delta\mathcal{X})$

$$\begin{aligned}
&\frac{1}{2}\int_{\mathcal{D}}\left[\frac{1}{\rho}(\star\Delta p_t)\wedge\Delta p_t + \frac{1}{I_\rho}(\star\Delta p_r)\wedge\Delta p_r + K(\star\Delta\epsilon_t)\wedge\ \Delta\epsilon_t + EI(\star\Delta\epsilon_r)\wedge\Delta\epsilon_r\right]\\
&+\frac{1}{2}K_{c1}\left(\int_{\mathcal{D}}\left[(k_3^1+k_1^1 l)\Delta p_t + k_1^1\Delta p_r + k_2^1\Delta\epsilon_t + (k_4^1 - k_2^1 l)\Delta\epsilon_r\right]\right)^2\\
&+\frac{1}{2}K_{c2}\left(\int_{\mathcal{D}}\left[(k_3^2+k_1^2 l)\Delta p_t + k_1^2\Delta p_r + k_2^2\Delta\epsilon_t + (k_4^2 - k_2^2 l)\Delta\epsilon_r\right]\right)^2\\
&+\frac{1}{2}\Delta p_c{}^T M_c{}^{-1}\Delta p_c
\end{aligned} \tag{16}$$

Stability Analysis

Using (16) and the Definition of The Norm (15), The Convexity Condition (14) Can Be Satisfied by Proper Choice of γ_1, γ_2 and α:

$$
\begin{aligned}
\alpha &= 2 \\
\gamma_1 &= \frac{1}{2}\min\left\{\frac{1}{\rho}, \frac{1}{I_\rho}, K, EI,\ \min\left\{\mathrm{eig}({M_c}^{-1})\right\}\right\} \\
\gamma_2 &= \tilde{\gamma_2}\cdot\max\Bigg\{8L\left[(k_3^1)^2 + (k_1^1)^2L + (k_3^2)^2 + (k_1^2)^2L\right] + 1, \\
&\qquad 4L\left[(k_1^1)^2 + (k_1^2)^2\right] + 1, 4L\left[(k_2^1)^2 + (k_2^2)^2\right] + 1, \\
&\qquad 8L\left[(k_4^1)^2 + (k_2^1)^2L + (k_4^2)^2 + (k_2^2)^2L\right] + 1\Bigg\}
\end{aligned}
$$

where

$$
\tilde{\gamma_2} = \frac{1}{2}\max\left\{\frac{1}{\rho}, \frac{1}{I_\rho}, K, EI,\ \max\left\{\mathrm{eig}({M_c}^{-1})\right\}, K_{c1}, K_{c2}\right\}
$$

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$$

The equilibrium configuration ($\mathcal{X}^*$) is Asymptotically Stable

Finite Dimensional Controller

Finite Dimensional Controller

Assumptions

- $q_c = \begin{bmatrix} x \\ \tilde{x} \end{bmatrix}$
- $p_c = \begin{bmatrix} M & 0 \\ 0 & \tilde{M} \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{\tilde{x}} \end{bmatrix}$
- $f_{c_2} = 0$

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Hamiltonians for The Cart and The Finite Dimensional Controller

$$\begin{aligned} H_{cart} &= \frac{1}{2} M \dot{x}^2 \\ H_{controller} &= \frac{1}{2} \tilde{M} \dot{\tilde{x}}^2 + \frac{1}{2} K_{c1} (x - \mathcal{C}_1)^2 \\ &\quad + \frac{1}{2} K_{c2} (\tilde{x} - \mathcal{C}_2)^2 \end{aligned}$$

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- **Controller Dynamics:**

$$\tilde{M} \ddot{\tilde{x}} = -K_{c2}(\tilde{x} - \mathcal{C}_2) - a_2 \dot{x} - a_3 \dot{\tilde{x}}$$

- **Cart Dynamics:**

$$M \ddot{x} = F$$

where

$$F = f_{c1} - K_{c1}(x - \mathcal{C}_1) - a_1 \dot{x} - a_2 \dot{\tilde{x}}$$

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THANK YOU