Trajectory Smoothing as a Linear Optimal Control Problem

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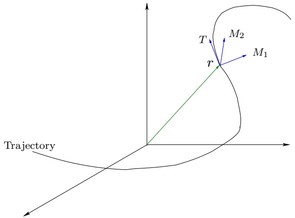
Background and Motivation

- To explore underlying strategies and motion (pursuit, collective motion etc.) governing control laws, by extracting parameters of motion (namely curvature, speed, lateral acceleration etc.) from sampled observations of trajectories.
- To extract control inputs from sampled data.

Generative Models for a Curve in $\mathbb{R}^3$ (Non-linear and Linear)

Natural Frenet Frame

$$\begin{aligned} \dot{r} &= \nu T \\ \dot{T} &= \nu\left(k_1 M_1 + k_2 M_2\right) \\ \dot{M}_1 &= -\nu k_1 T \\ \dot{M}_2 &= -\nu k_2 T \end{aligned} \tag{1}$$



- The natural curvatures are the steering inputs and the speed is a time function dictated by propulsive/lift/drag mechanisms.

Linear Generative Model

$$\begin{aligned} \dot{r} &= v \\ \dot{v} &= a \\ \dot{a} &= u \end{aligned} \tag{2}$$

- Jerk, i.e. the third-derivative of position, is viewed as the control.

LTI representation

$$\begin{aligned} \dot{x} &= Ax + Bu \\ r &= Cx \end{aligned} \tag{3}$$

with,

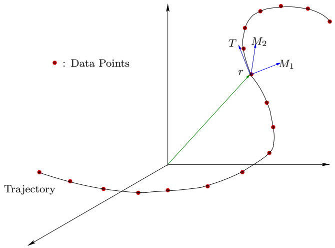
$$x = \begin{bmatrix} r^T & v^T & a^T \end{bmatrix}^T;$$

$$A = \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{bmatrix}; B = \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix};$$

$$C = \begin{bmatrix} I & 0 & 0 \end{bmatrix}$$

- Controllable and Observable

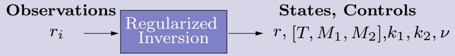
Regularized Inverse Problem



- Given a time series of observed positions, generate a smooth trajectory to fit the data points.
- The inverse problem is ill-posed.
  - Highly sensitive to noise.
  - Non-unique.
- A regularization parameter is introduced to control the amount of smoothing.
- *Ordinary cross validation* is a standard approach to choose an optimal value for the regularization parameter.

Extracting Curvature (Inverse Problem)

Non-linear Optimization



$$\text{Minimize} \quad \left(\sum_{i=0}^{N} \|r(t_i) - r_i\|^2 + \lambda \int_0^T (\dot{k}_1^2 + \dot{k}_2^2 + \dot{\nu}^2) dt \right) \tag{4}$$

subject to Dynamics in (1), Initial Condition, and Input

Linear-Quadratic Control

$$\text{Minimize} \quad \left(\sum_{i=0}^{N} \|r(t_i) - r_i\|^2 + \lambda \int_0^T u^T u dt \right) \tag{5}$$

subject to Dynamics in (3), Initial Condition, and Input

Relationship between Two Approaches for Modelling a Curve

Natural-Frenet Frame → Linear Model (Triple Integrator)

$$v = \nu T$$

$$a = \dot{\nu} T + \nu^2 k_1 M_1 + \nu^2 k_2 M_2$$

$$u = (\ddot{\nu} - \nu^3 (k_1^2 + k_2^2)) T + (3\nu\dot{\nu} k_1 + \nu^2 \dot{k}_1) M_1 + (3\nu\dot{\nu} k_2 + \nu^2 \dot{k}_2) M_2$$

Linear Model (Triple Integrator) → Natural-Frenet Frame

$$\nu = \|v\|$$

$$T = \frac{v}{\|v\|}$$

$$\dot{T} = \frac{1}{\nu}\left(a - (a \cdot T) T\right)$$

$$\kappa = \frac{\|\dot{T}\|}{\nu}$$

$$\tau = \frac{v \cdot (a \times u)}{\|v \times a\|^2}$$

- k_1, k_2, M_1, M_2 can be computed by assuming suitable intial conditions.

$$k_1(t) = \kappa \cos\left(\theta_0 + \int_0^t \tau(\sigma) d\sigma\right)$$

$$k_2(t) = \kappa \sin\left(\theta_0 + \int_0^t \tau(\sigma) d\sigma\right)$$

$$M_1(t) = M_1(0) - \int_0^t \nu(\sigma) k_1(\sigma) T(\sigma) d\sigma$$

$$M_2(t) = M_2(0) - \int_0^t \nu(\sigma) k_2(\sigma) T(\sigma) d\sigma$$

Application of Path Independence Lemma

- **Optimal Control Problem**:

$$\begin{aligned} \underset{x(t_0),u}{\text{Minimize}} \quad & J(x(t_0),u) = \sum_{i=0}^{N} \|r(t_i) - r_i\|^2 + \lambda \int_0^T u^T u dt \\ \text{subject to} \quad & x(t_0) \in \mathbb{R}^n, \\ & u \in \mathcal{U}, \\ & \text{Dynamics in (3)} \end{aligned} \tag{6}$$

- **Path Independence**:
  Along trajectories of (3)

$$\begin{aligned} 0 &= x^T(t_i)K(t_i^+)x(t_i) - x^T(t_{i+1})K(t_{i+1}^-)x(t_{i+1}) \\ &\quad + \int_{t_i^+}^{t_{i+1}^-} \begin{bmatrix} x \\ u \end{bmatrix}^T \begin{bmatrix} \dot{K} + A^T K + KA & KB \\ B^T K & 0 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} dt \end{aligned}$$

$$0 = x^T(t_i)\eta(t_i^+) - x^T(t_{i+1})\eta(t_{i+1}^-) + \int_{t_i^+}^{t_{i+1}^-} \begin{bmatrix} x \\ u \end{bmatrix}^T \begin{bmatrix} \dot{\eta} + A^T \eta \\ B^T \eta \end{bmatrix} dt$$

for all $i \in \{0, 1, \cdots, N-1\}$

Application of Path Independence Lemma

- **Assumptions on the the dynamics and boundary values of K and η:**

$$\begin{aligned} \dot{K} &= -A^T K - KA + KBB^T K, \\ K(t_N^+) &= 0, \\ K(t_i^+) - K(t_i^-) &= -\tfrac{1}{\lambda} C^T C. \end{aligned} \quad (7)$$

$$\begin{aligned} \dot{\eta} &= -\left(A^T - KBB^T\right)\eta, \\ \eta(t_N^+) &= 0, \\ \eta(t_i^+) - \eta(t_i^-) &= \tfrac{2}{\lambda} C^T r_i. \end{aligned} \quad (8)$$

- With the assumptions (7) and (8), we obtain

$$J(x(t_0), u) = \lambda \left[x^T(t_0) K(t_0^-) x(t_0) + x^T(t_0)\eta(t_0^-) \right] + \sum_{i=0}^{N} r_i^T r_i - \frac{1}{4}\lambda \int_0^T \|B^T \eta(t)\|^2 dt$$
$$+ \lambda \int_0^T \|u(t) + B^T \left(K(t)x(t) + \frac{1}{2}\eta(t) \right) \|^2 dt. \quad (9)$$

- **Optimal control input:**

$$u_{opt}(t) = -B^T \left(K(t)x(t) + \frac{1}{2}\eta(t) \right) \quad (10)$$

- **Optimal initial condition:**

$$\left[K(t_0^-) \right] x_{opt}(t_0) + \frac{1}{2}\eta(t_0^-) = 0. \quad (11)$$

Existence of Solution for (11) - Sketch of Proof

Proposition 1

The solution of the Riccati equation (7) assumes the form

$$K(t_i^-) = \frac{1}{\lambda} \sum_{k=i}^{N} \Phi_\Sigma(t_i, t_k) C^T C \Phi_\Sigma^T(t_i, t_k)$$

for any $i \in \{0, 1, \cdots, N\}$ where $\Sigma(t) = -(A - \frac{1}{2} B B^T K(t))^T$ and Φ_Σ is the transition matrix of Σ.

- Holds true for $i = N$.
- Apply mathematical induction.

Proposition 2

$(-\Sigma^T, C)$ forms an observable pair for the problem of our interest (3).

- Apply Silverman-Meadows rank condition.

Existence of Solution for (11) - Sketch of Proof

Theorem 1

The equation

$$\left[K(t_0^-)\right] x_{opt}(t_0) + \frac{1}{2}\eta(t_0^-) = 0.$$

is uniquely solvable for almost any time index set $\{t_i\}_{i=0}^N$.

- Observe $K(t_0^-)$ can be represented as $K(t_0^-) = \frac{1}{\lambda}\mathfrak{C}^T\mathfrak{C}$, with

$$\mathfrak{C} = \begin{bmatrix} C \\ C\Phi_{-\Sigma^T}(t_1, t_0) \\ \vdots \\ C\Phi_{-\Sigma^T}(t_N, t_0) \end{bmatrix}.$$

- Consider the system $\dot{\xi} = -\Sigma^T\xi$; $\gamma = C\xi$. The outputs, corresponding to two different initial conditions, do not match identically over any interval.
- $$\xi_a \neq \xi_b \quad \Rightarrow \quad \mathfrak{C}\xi_a \neq \mathfrak{C}\xi_b \qquad \text{(almost surely)}$$
- Otherwise, consider an arbitrary close perturbation of the original time index set $\{t_i\}_{i=0}^N$, to obtain full rank for $\mathfrak{C}$.

Linearity in the Reconstructed Trajectory

- Closed loop dynamics:

$$\dot{x}(t) = -\tilde{\Sigma}^T x(t) - \frac{1}{2} BB^T \eta(t)$$

with $\tilde{\Sigma} = \left[A - BB^T K(t)\right]^T$.

- $x_{opt}(t_0)$ and $\eta(\cdot)$ are linear in observed data $\{r_i\}_{i=0}^N$.

$$r(t_k) = \frac{1}{\lambda} \sum_{i=0}^{N} \left[C \mathcal{F}_\lambda(k,i) C^T\right] r_i \tag{12}$$

where

$$\mathcal{F}_\lambda(k,i) = \Phi_{\tilde{\Sigma}}^T(t_0, t_k) \left[K(t_0^-)\right]^{-1} \Phi_{\tilde{\Sigma}}(t_0, t_i) + \sum_{j=1}^{\min\{i,k\}} \left(\int_{t_{j-1}}^{t_j} \Phi_{\tilde{\Sigma}}^T(\sigma, t_k) BB^T \Phi_{\tilde{\Sigma}}(\sigma, t_i) d\sigma \right)$$

- Can be be viewed as a global alternative to Savitzky-Golay smoothing filters.
- Can be used as a building block to obtain a fixed lag smoothing algorithm.

An Alternative Co-State Based Approach

- Co-state variables:

$$p(t) \triangleq K(t)x(t) + \frac{1}{2}\eta(t)$$

- An optimal trajectory between two observation times can be viewed as the base integral curve of the following Hamiltonian dynamics

$$\frac{d}{dt}\left[\begin{array}{c} x(t) \\ p(t) \end{array}\right] = \left[\begin{array}{cc} A & -BB^T \\ 0 & -A^T \end{array}\right]\left[\begin{array}{c} x(t) \\ p(t) \end{array}\right]$$

- Jump condition for the co-state variables:

$$p(t_i^+) - p(t_i^-) = \frac{1}{\lambda}C^T\left(r_i - r(t_i)\right)$$

- Terminal condition for the co-state variables:

$$p(t_N^+) = 0$$
$$p(t_0^-) = 0$$

An Alternative Co-State Based Approach

- Forward-propagation of $x(t_i)$ and $p(t_i^+)$:

$$\begin{bmatrix} x(t_{i+1}) \\ p(t_{i+1}^+) \end{bmatrix} = \begin{bmatrix} e^{A\Delta_i} & -e^{A\Delta_i}\mathbb{W}_i \\ -\frac{1}{\lambda}C^TCe^{A\Delta_i} & \left[e^{-A^T\Delta_i} + \frac{1}{\lambda}C^TCe^{A\Delta_i}\mathbb{W}_i\right] \end{bmatrix} \begin{bmatrix} x(t_i) \\ p(t_i^+) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{\lambda}C^T \end{bmatrix} r_{i+1}$$

where $\mathbb{W}_i$ is defined as

$$\mathbb{W}_i = \int_0^{\Delta_i} e^{-A\sigma}BB^Te^{-A^T\sigma}d\sigma \quad (\Delta_i = t_{i+1} - t_i)$$

- **Optimal initial condition** is obtained by solving

$$[0 \;\; I]\left(\prod_{i=0}^{N-1}\Lambda_i\right)\begin{bmatrix} I \\ -\frac{1}{\lambda}C^TC \end{bmatrix} x(t_0) = -[0 \;\; I]\sum_{i=0}^{N}\left(\prod_{j=i}^{N-1}\Lambda_j\right)\Gamma r_i \qquad (13)$$

where,

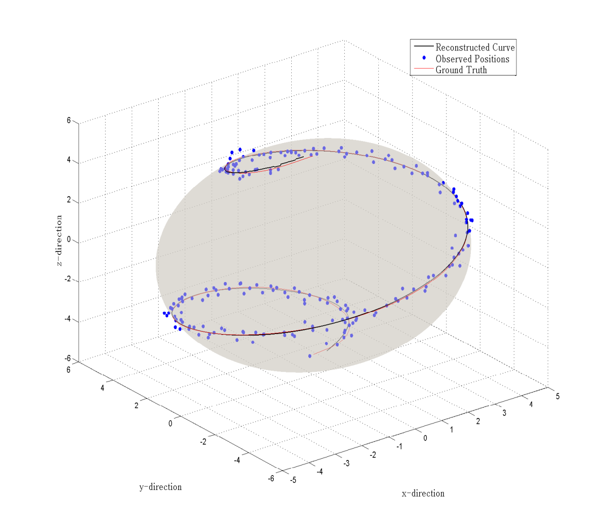
$$\Lambda_i = \begin{bmatrix} e^{A\Delta_i} & -e^{A\Delta_i}\mathbb{W}_i \\ -\frac{1}{\lambda}C^TCe^{A\Delta_i} & \left[e^{-A^T\Delta_i} + \frac{1}{\lambda}C^TCe^{A\Delta_i}\mathbb{W}_i\right] \end{bmatrix}; \Gamma = \begin{bmatrix} 0 \\ \frac{1}{\lambda}C^T \end{bmatrix}$$

Cross-validation Approach to Determination of Penalty Parameter

- We use "leaving-out-one" version of the Ordinary Cross Validation (OCV) technique.

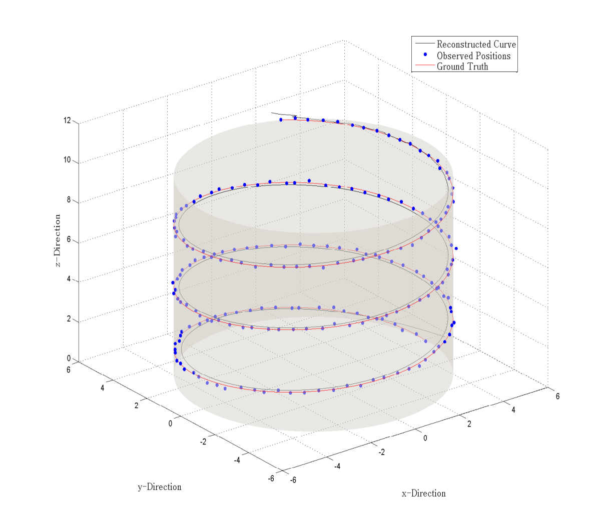
- Let, $\{x_{opt}^{[\lambda,k]}, u^{[\lambda,k]}\}$ be a minimizer of:
$$\sum_{\substack{i=0\\ i\neq k}}^{N} \|r(t_i) - r_i\|^2 + \lambda \int_0^T u^T u dt$$
- Let the reconstructed trajectory be $r^{[\lambda,k]}(\cdot)$.
- Then the **OCV cost** is defined as:
$$V_0(\lambda) = \frac{1}{N+1} \sum_{k=0}^{N} \|r^{[\lambda,k]}(t_k) - r_k\|^2$$
- Hence, **OCV estimate** for λ is defined as:
$$\lambda^* = \underset{\lambda \in \mathbb{R}_+}{\operatorname{argmin}} \; V_0(\lambda)$$

Numerical Result - Spherical Curve



Avg. Fit Error/Radius: 13.686×10^{-3}.

Numerical Result - Circular Helix



Avg. Fit Error/Radius: 12.346×10^{-3}.

References

- E. Justh and P. S. Krishnaprasad, “Optimal Natural Frames”, Comm. Inf. Syst., 11(1):17-34, 2011.
- T. Flash and N. Hogan, “The coordination of arm movements: An experimentally confirmed mathematical model”, The Journal of Neuroscience, 5(7):1688-1703, 1985.
- R. L. Bishop, “There is more than one way to frame a curve”, The American Mathematical Monthly, 82(3):246-251, 1975.
- L. M. Silverman and H. E. Meadows, “Controllability and observability in time-variable linear systems”, SIAM J. on Control and Optimization, 5(1):64-73, 1967.
- P.V. Reddy, *Steering laws for pursuit*, M. S. Thesis, University of Maryland, College Park, 2007.
- B. Dey and P. S. Krishnaprasad, “Trajectory smoothing as a linear optimal control problem (extended version)”, http://www.isr.umd.edu/Labs/ISL/SMOOTHING.

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