Control-Theoretic Data Smoothing

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- **Objective**: Given a time series of noisy data, reconstruct/generate a path in the underlying space which traverses through the data points.
  - Curve Reconstruction
  - Quantum Information Processing (Quantum State Traversal) **Brody et al., PRL 2012**
  - Computer Vision (Curve Completion) **Ben-Yosef et al., PAMI 2012**

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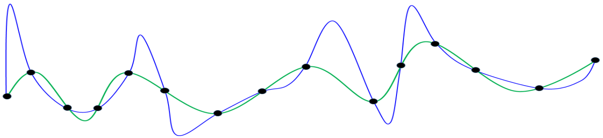
  - Non-unique
  - High sensitivity to noise



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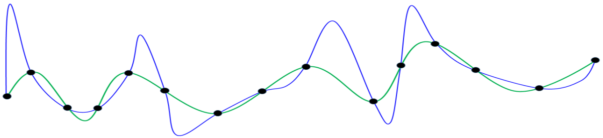

- **Our Approach**: **Regularized Inversion**
  - Introduce a generative model to treat the data points as output from an underlying dynamical system.
  - Impose regularization by adding a penalty term to fit error.

Measured Data

$\{r_i\}_{i=0}^{N}$ → **Regularized Inversion** → Smoothened Data $\mathbf{r}(\cdot)$

Generative Model

$$\dot{x}(t) = Ax(t) + Bu(t)$$

or,

$$\dot{q}(t) = f\big(t, q(t), u(t)\big)$$

or,

$$\dot{g}(t) = g(t)\xi_u(t)$$

Cost

$$\underset{q(t_0);u(\cdot)}{\text{Minimize}} \quad \left(\sum_{i=0}^{N} d^2\Big(\mathbf{r}(t_i), r_i\Big) + \lambda \int_{t_0}^{t_N} L\big(u(t)\big)\,dt \right)$$

Outline

1. Data Smoothing in a Euclidean Setting ($\mathbb{R}^n$)
   - Maximum Principle
   - Sketch of Proof
   - Example Problem
2. Data Smoothing in Matrix Lie-Group Setting (G)
   - Maximum Principle
   - Sketch of Proof
   - Lie-Poisson Reduction
   - Example Problem
3. Conclusion

Maximum Principle for Data Smoothing on $\mathbb{R}^n$

Data Smoothing as a Regularized Inversion

$$\underset{q(t_0);u}{\text{Minimize}} \quad J\big(q(t_0),u\big) = \int_{t_0}^{t_N} L\big(t,q(t),u(t)\big)dt + \sum_{i=0}^{N} F_i\big(q(t_i)\big) \tag{1}$$

$$\text{subject to:} \quad \dot{q}(t) = f\big(t,q(t),u(t)\big), \quad q:[t_0,t_N]\to\mathbb{R}^n, \quad u\in\mathcal{U}=\big\{u:[t_0,t_N]\to U\big\}$$

PMP for data smoothing (Theorem 2.2)

Let u^* be an optimal control input for (1), and q^* denote the corresponding state trajectory. Then, by defining a pre-Hamiltonian as $H(t,q,p,u) = \langle p, f(t,q,u)\rangle - L(t,q,u)$, we can show that there exists a costate trajectory $p:[t_0,t_N]\to\mathbb{R}^n$ such that

$$\dot{q}^*(t) = \frac{\partial H}{\partial p}\big(t,q^*(t),p(t),u^*(t)\big), \qquad \dot{p}(t) = -\frac{\partial H}{\partial q}\big(t,q^*(t),p(t),u^*(t)\big), \tag{2}$$

$$\text{and,} \qquad H(t,q^*,p,u^*) = \max_{u\in U} H(t,q^*,p,u), \tag{3}$$

at the points of continuity. Moreover, the penalties on intermediate state yield jump discontinuities given by

$$p(t_i^+) - p(t_i^-) = \frac{\partial F_i\big(q(t_i)\big)}{\partial q(t_i)}, \qquad i=0,1,\ldots,N. \tag{4}$$

Also, boundary values of the costate variables satisfy

$$p(t_0^-) = p(t_N^+) = 0. \tag{5}$$

Highlights of The Proof

- We introduce a **new state variable**: $\tilde{q} : [t_0, t_N] \to \mathbb{R}$.

$$y(t) \triangleq \begin{pmatrix} \tilde{q}(t) \\ q(t) \end{pmatrix} \in \mathbb{R}^{n+1} \quad \Longrightarrow \quad \dot{y}(t) = \underbrace{\begin{pmatrix} L\big(t, q(t), u(t)\big) \\ f\big(t, q(t), u(t)\big) \end{pmatrix}}_{\triangleq g\left(t, y(t), u(t)\right)}, \quad y(t_i^+) - y(t_i^-) = \begin{pmatrix} F_i\big(q(t_i)\big) \\ 0 \end{pmatrix}$$

- This transforms the problem into the **Mayer form**, as $J\big(q(t_0), u\big) = \tilde{q}(t_N^+) = J\big(y(t_0), u\big)$.
- **Perturbed Control** (Needle Variation):

$$u_{w,I}(t) \triangleq \begin{cases} u^*(t) & \text{if} \quad t \notin I \\ w & \text{if} \quad t \in I \end{cases}, \qquad w \in U, \quad I = (b - \epsilon a, b] \subset (t_0, t_N), a > 0$$

- Construct the **perturbed trajectory**, and compute the perturbation in the terminal state $y(t_N^+)$.
- Construct the **terminal cone** at $y^*(t_N^+)$, through concatenation of needle variations.

Example Problem: Trajectory Reconstruction[1]

Trajectory Reconstruction

$$\underset{q(t_0),u}{\text{Minimize}} \quad J\big(q(t_0),u\big) = \sum_{i=0}^{N} \|\mathbf{r}(t_i) - r_i\|^2 + \lambda \int_{t_0}^{t_N} u^T(t)u(t)dt$$

$$\text{subject to} \quad \dot{q}(t) = Aq(t) + Bu(t), \quad \mathbf{r}(t) = Cq(t)$$

$$q(t_0) \in \mathbb{R}^9, \quad u \in \mathcal{U} \tag{6}$$

$$A = \begin{bmatrix} 0 & \mathbb{I}_3 & 0 \\ 0 & 0 & \mathbb{I}_3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 0 \\ \mathbb{I}_3 \end{bmatrix}$$

$$C = \begin{bmatrix} \mathbb{I}_3 & 0 & 0 \end{bmatrix}$$

[1] B. Dey, P. S. Krishnaprasad, "Trajectory Smoothing as a Linear Optimal Control Problem", 50th Annual Allerton Conf., *pp. 1490 - 1497*, Oct'2012.

Example Problem: Trajectory Reconstruction[1]

Trajectory Reconstruction

$$\begin{aligned} \underset{q(t_0),u}{\text{Minimize}} \quad & J\big(q(t_0),u\big) = \sum_{i=0}^{N} \|\mathbf{r}(t_i) - r_i\|^2 + \lambda \int_{t_0}^{t_N} u^T(t)u(t)dt \\ \text{subject to} \quad & \dot{q}(t) = Aq(t) + Bu(t), \quad \mathbf{r}(t) = Cq(t) \\ & q(t_0) \in \mathbb{R}^9, \quad u \in \mathcal{U} \end{aligned} \tag{6}$$

$$A = \begin{bmatrix} 0 & \mathbb{I}_3 & 0 \\ 0 & 0 & \mathbb{I}_3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 0 \\ \mathbb{I}_3 \end{bmatrix}$$

$$C = \begin{bmatrix} \mathbb{I}_3 & 0 & 0 \end{bmatrix}$$

- $L(t,q,u) = \lambda u^T u$
- $F_i\big(q(t_i)\big) = q(t_i)C^T Cq(t_i) - 2q(t_i)C^T r_i + r_i^T r_i$
- **Optimal Control Input**

$$u^*(t) = \frac{1}{2\lambda} B^T p(t)$$

- **State-costate Dynamics**

$$\frac{d}{dt}\begin{bmatrix} q^*(t) \\ p(t) \end{bmatrix} = \begin{bmatrix} A & \frac{1}{2\lambda}BB^T \\ 0 & -A^T \end{bmatrix}\begin{bmatrix} q^*(t) \\ p(t) \end{bmatrix}$$

- **Boundary Values and Jump Conditions**

$$\begin{aligned} p(t_0^-) &= p(t_N^+) = 0 \\ p(t_i^+) - p(t_i^-) &= 2C^T\left[Cq(t_i) - r_i\right] \end{aligned}$$

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Maximum Principle for Data Smoothing in a Matrix Lie-Group Setting

Data Smoothing as an Optimal Control Problem on Lie Group (G)

$$\underset{g(t_0);u}{\text{Minimize}} \quad J(g(t_0),u) = \int_{t_0}^{t_N} L(u(t))dt + \sum_{i=0}^{N} F(g(t_i), g_i) \tag{7}$$

$$\text{subject to:} \quad \dot{g}(t) = g(t)\xi_u(t) = T_e L_{g(t)} \cdot \xi_u(t), \quad g : [t_0, t_N] \to G, \quad u \in \mathcal{U} = \{u : [t_0, t_N] \to U\}$$

PMP for data smoothing (Theorem 3.2)

Let u^* be a solution for the optimal control problem (7). The corresponding state trajectory g^* is the *base integral curve* of a Hamiltonian vector field $X_{H(g^*,p,u^*)}$ on T^*G, where the pre-Hamiltonian is defined as

$$H(g,p,u) = \langle p, T_e L_g \cdot \xi_u \rangle - L(u) \tag{8}$$

and the optimal control input maximizes H, i.e.

$$H(g^*,p,u^*) = \underset{u \in U}{\text{Max}}\, H(g^*,p,u). \tag{9}$$

(Observe that the pre-Hamiltonian is G invariant.) Moreover, data dependency of the cost functional causes jump discontinuities in p, and the corresponding boundary values and jump conditions are given as

$$p(t_0^-) = p(t_N^+) = 0 \tag{10}$$

$$\text{and,} \qquad p(t_i^+) - p(t_i^-) = D_{g^*(t_i)}F, \qquad i = 0, 1, \cdots, N \tag{11}$$

where $D_{g^*(t_i)}F$ represents the *Frechet derivative* of the fit-error at $g^*(t_i) \in G$.

Highlights of The Proof

- We use a **variational approach**.
- **Express Cost in terms of The Hamiltonian**:

$$J\big(g(t_0), u\big) = \int_{t_0}^{t_N} \Big(\langle p(t), T_e L_{g(t)} \cdot \xi_u(t)\rangle - H\big(g(t), p(t), u(t)\big)\Big) dt + \sum_{i=0}^{N} F\big(g(t_i), g_i\big)$$

- **Perturbed Control**:

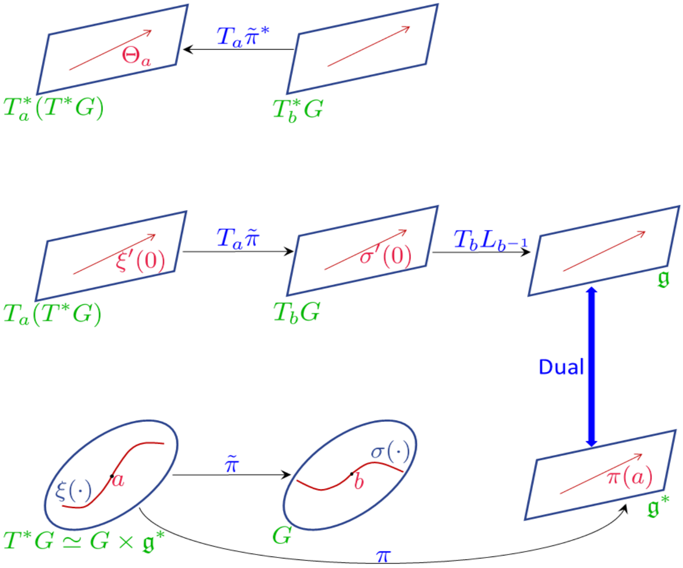
$$u_\epsilon = u^* + \epsilon\delta u, \;\; \epsilon > 0 \qquad \Longrightarrow \qquad \xi_\epsilon = \xi_{u^*} + \epsilon\delta\xi_u$$

- **Perturbation in State Trajectory**:

$$g_\epsilon = g^* + \epsilon\delta g + O(\epsilon^2), \qquad \text{where} \qquad \delta g = g^*\delta\xi_u$$

- **Invoke first order necessary condition**, i.e., $\delta J\big(g^*(t_0), u^*\big) = 0$.
- **Invoke second order necessary condition**, i.e., $\delta^2 J\big(g^*(t_0), u^*\big) \geq 0$.

A Quick Review of Lie-Poisson Reduction



- **Poincare 1-form**:

$$\Theta_a\Big(\xi'(0)\Big) = \Big\langle \pi\big(\xi(0)\big), T_{\tilde{\pi}(a)}L_{\tilde{\pi}(a)^{-1}} \cdot \big(T_a\tilde{\pi} \cdot \xi'(0)\big)\Big\rangle$$

- Define a Hamiltonian vector field on T^*G ($H_\bullet$), by exploiting the symplectic form associated with the Poincare 1-form ($\omega = -d\Theta$).
- **Poisson Bracket**:

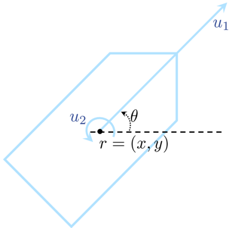
$$\phi, \psi \mapsto \{\phi, \psi\} = \omega(\mathrm{H}_\phi, \mathrm{H}_\psi)$$

  - ϕ, ψ are smooth ($\mathfrak{C}^\infty$) functions on T^*G
- **Lie-Poisson Bracket**:

$$\pi^*\{h_1, h_2\}_{\mathfrak{g}^*} = \{h_1, h_2\}_{\mathfrak{g}^*} \circ \pi = \{\pi^* h_1, \pi^* h_2\}$$

  - π^*: Pullback by π
  - h_1, h_2 are smooth ($\mathfrak{C}^\infty$) functions on $\mathfrak{g}^*$

Example Problem: Data Smoothing on $SE(2)$



Dynamics

$$\dot{x} = u_1 \cos\theta$$
$$\dot{y} = u_1 \sin\theta$$
$$\dot{\theta} = u_2$$

Lie-Group Formulation

$$\dot{g} = g\xi_u = g\big(u_2 X_1 + u_1 X_2\big), \qquad g \in SE(2);\;\; X_1, X_2 \in \mathfrak{se}(2)$$

where,

$$g = \begin{bmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and,} \quad \begin{array}{rcl} X_1 & = & [e_2, -e_1, 0_{3\times 1}] \\ X_2 & = & [0_{3\times 2}, e_1] \\ X_3 & = & [0_{3\times 2}, e_2] \end{array}$$

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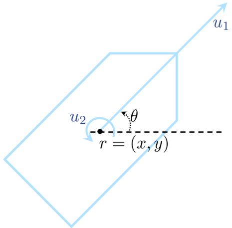

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- Find a curve $g : [t_0, t_N] \to SE(2)$, to traverse through targeted positions $r_0 \to r_1 \to \cdots \to r_N$

$$\underset{g(t_0), u_1, u_2}{\text{Minimize}} \quad \sum_{i=0}^{N} \|r(t_i) - r_i\|^2 + \lambda \int_{t_0}^{t_N} (u_1^2 + u_2^2)\, dt$$
$$\text{subject to} \quad g(t_0) \in SE(2), \quad u_1, u_2 \in \mathcal{U},$$
$$\dot{g} = g\big(u_2 X_1 + u_1 X_2\big), \tag{12}$$

- **Lagrangian**: $L(u) = \lambda(u_1^2 + u_2^2) = \lambda\langle \xi_u, \xi_u\rangle_{\mathfrak{se}(2)}$, where $\langle v_1, v_2\rangle_{\mathfrak{se}(2)} = \mathrm{Tr}(v_1 M v_2^T)$, $M = diag\{\frac{1}{2}, \frac{1}{2}, 1\}$
- **Intermediate State-Cost**: $F(g(t_i), r_i) = \|A g(t_i) e_3 - r_i\|^2$, where $A = [e_1 \;\; e_2]^T$

Example Problem: Data Smoothing on $SE(2)$ [Contd.]

- **Introduce**: $\mu = \sum_{i=1}^{3} \mu_i X_i^\flat \in \mathfrak{se}^*(2)$, where $\langle X_i, X_j^\flat \rangle = \begin{cases} 1 & \text{if, } i = j \\ 0 & \text{otherwise} \end{cases} \quad i, j \in \{1, 2, 3\}$
- **pre-Hamiltonian**: $H(g, p, u) = \langle p, T_e L_g \cdot \xi_u \rangle - L(u) = \langle T_e L_g^* \cdot p, \xi_u \rangle - L(u) = u_2 \mu_1 + u_1 \mu_2 - \lambda(u_1^2 + u_2^2).$

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- **Optimal Control Input** $\begin{pmatrix} u_1^* \\ u_2^* \end{pmatrix} = \frac{1}{2\lambda} \begin{pmatrix} \mu_2 \\ \mu_1 \end{pmatrix}$
- **Reduced Costate Dynamics** $\begin{pmatrix} \dot{\mu}_1 \\ \dot{\mu}_2 \\ \dot{\mu}_3 \end{pmatrix} = \frac{1}{2\lambda} \begin{pmatrix} -\mu_2\mu_3 \\ \mu_3\mu_1 \\ -\mu_1\mu_2 \end{pmatrix} \qquad t \in (t_k, t_{k+1})$
- **Jump Conditions** $\mu_i(t_k^+) - \mu_i(t_k^-) = \mathrm{Tr}\left(2g(t_k)^T A^T \left[Ag(t_k)e_3 - r_k\right] e_3^T X_i^T\right) \qquad k \in \{0, 1, \cdots, N-1\}$
- **Boundary Values** $\mu_i(t_0^-) = \mu_i(t_N^+) = 0$

Example Problem: Data Smoothing on $SE(2)$ [Contd.]

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- **Optimal Control Input** $\begin{pmatrix} u_1^* \\ u_2^* \end{pmatrix} = \frac{1}{2\lambda} \begin{pmatrix} \mu_2 \\ \mu_1 \end{pmatrix}$
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- **Boundary Values** $\mu_i(t_0^-) = \mu_i(t_N^+) = 0$
- **Conserved Quantities** $\begin{cases} \text{Hamiltonian:} & h = \frac{1}{4\lambda}(\mu_1^2 + \mu_2^2) \\ \text{Casimir:} & C = \frac{1}{4\lambda}(\mu_2^2 + \mu_3^2) \end{cases}$
- **Closed-form Solution**

$$\begin{cases} \mu_1(t) &= \pm 2\sqrt{\lambda h}\, \mathrm{Cn}\left(\sqrt{\frac{C}{\lambda}}(t + \phi_k), \sqrt{\frac{h}{C}}\right) \\ \mu_2(t) &= 2\sqrt{\lambda h}\, \mathrm{Sn}\left(\sqrt{\frac{C}{\lambda}}(t + \phi_k), \sqrt{\frac{h}{C}}\right) \\ \mu_3(t) &= \pm 2\sqrt{\lambda C}\, \mathrm{Dn}\left(\sqrt{\frac{C}{\lambda}}(t + \phi_k), \sqrt{\frac{h}{C}}\right) \end{cases} \qquad t \in (t_k, t_{k+1})$$

Conclusion

- **Summary**
  - Developed an extended version of the maximum principle to address data smoothing using generative models.
  - Results are applicable to problems in both Euclidean and finite dimensional matrix Lie group settings.
  - This approach yields solution in a semi-analytical way.

- **Future Directions**
  - To consider Lagrangians involving higher derivatives of control input.

References

- B. Dey and P. S. Krishnaprasad, "Control-Theoretic Data Smoothing", Proc. *53rd IEEE Conference on Decision and Control*, pp. 5064-5070, Los Angeles, CA, 2014.
- B. Dey and P. S. Krishnaprasad, "Trajectory Smoothing as a Linear Optimal Control Problem", Proc. *50th Annual Allerton Conference on Communication, Control, and Computing*, pp. 1490-1497, Allerton, IL, 2012.
- E. Justh and P. S. Krishnaprasad, "Optimal Natural Frames", Communications in Information and Systems, 11(1):17-34, 2011.
- P. S. Krishnaprasad, "Lie-Poisson structures, dual-spin spacecraft and asymptotic stability", Nonlinear Analysis: Theory, Methods and Applications, 9(10):1011-1035, 1985.
- P. S. Krishnaprasad, "Optimal control and Poisson reduction", *Tech. Rep. TR 93-87*, Institute for Systems Research, University of Maryland, College Park, MD, 1993.
- H. Sussmann and J. Willems, "300 years of optimal control: from the brachystochrone to the maximum principle", IEEE Control Systems, 17(3):32-44, 1997.
- D. Liberzon, *Calculus of Variations and Optimal Control Theory: A Concise Introduction*, Princeton, NJ: Princeton University Press, 2011.
- G. B. Yosef and O. B. Shahar, "A tangent bundle theory for visual curve compeltion", IEEE Trans. on Pattern Analysis and Machine Intelligence, 34(7):1263-1280, 2012.
- D. C. Brody, D. D. Holm and D. M. Meier, "Quantum Splines", Physical Review Letter, 109(10):100501, 2012.

THANK YOU !!!