Stability and Pure Shape Equilibria
for
Beacon-referenced Cyclic Pursuit

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Motivation

Some tasks/missions can be accomplished more effectively by groups/collectives of autonomous agents

- Environmental Monitoring
- Search & Rescue
- Intelligence, Surveillance & Reconnaissance (ISR)

Collective control uses local/pairwise interactions to generate desired global motions

- Efficiency
- Robustness

ONR's demonstration of prototype tube-launched UAVs in April 2015.
Snapshot from: https://www.youtube.com/watch?v=8FukTsKmXOo

ONR's demonstration of CARACaS (Control Architecture for Robotic Agent Command and Sensing) in August 2014.
http://www.onr.navy.mil/Media-Center/Press-Releases/2014/ autonomous-swarm-boat-unmanned-caracas.aspx

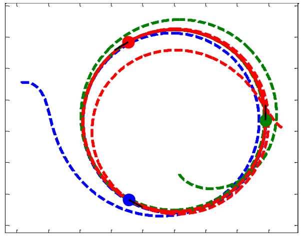
Pursuit Based Collectives

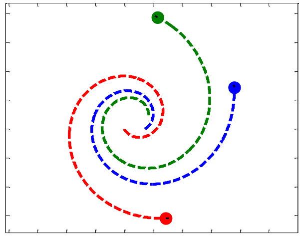
- Pursuit interactions as a building block for collective control

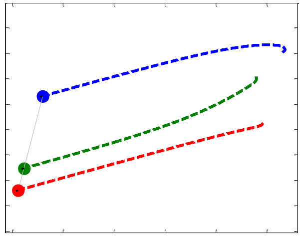
UW-Milwaukee Bio Sciences

STARFLAG project

- Cyclic Constant Bearing (CB) pursuit can generate circling, spiraling & rectilinear motions
  - Limitation: Location and Size of the formation depends on initial conditions







- Galloway, Justh, Krishnaprasad, "Symmetry and Reduction in Collectives: Cyclic Pursuit Strategies", Proc. R. Soc. A, 2013.

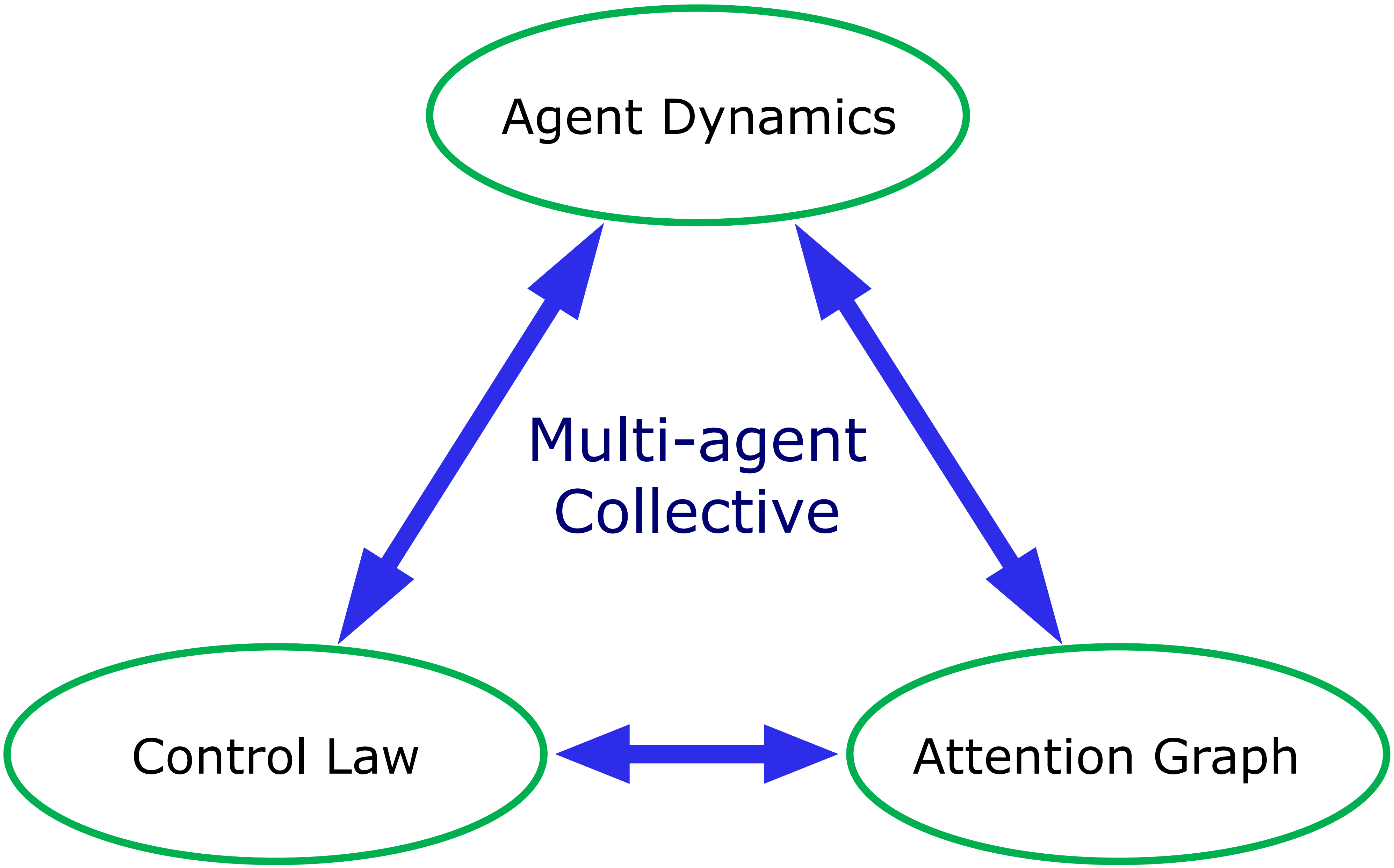
- Later work introduced an external reference (i.e. a beacon) based framework
  - Can represent a distress signal, resource peak, region of interest, etc.

- Galloway, Dey, "Station Keeping through Beacon-referenced Cyclic Pursuit", ACC 2015.

Overview

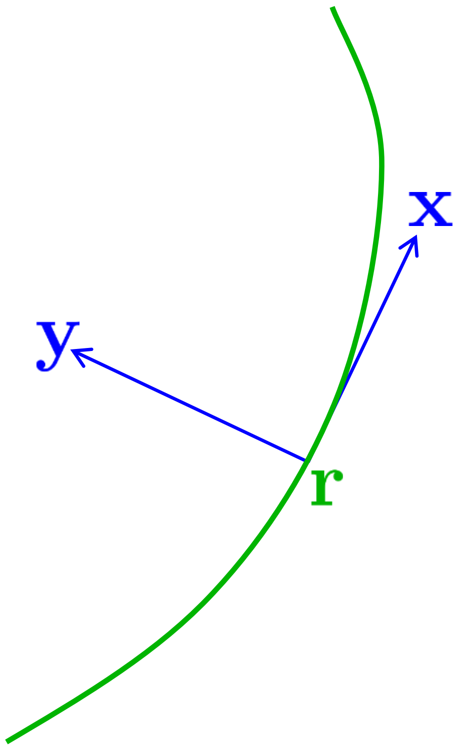
- Modeling
- Reduction to Shape Space
- Local Stability of Relative Equilibria
- Separation of Size and Pure Shape
- Pure Shape Equilibria
- Future Directions

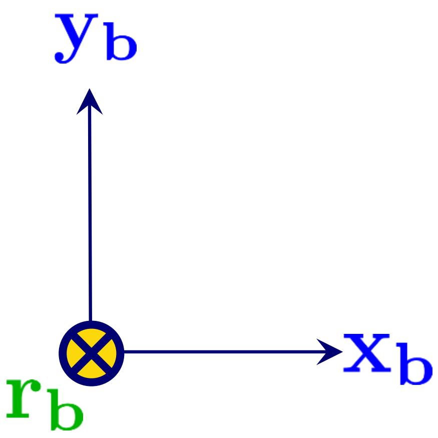
Key Ingredients for a Multi-agent Collective (MaC)



Modeling: Moving Agents

- Position vector: $\mathbf{r}$
- Natural Frenet frame: $[\mathbf{x} \;\; \mathbf{y}]$
- Unit tangent vector: $\mathbf{x} = \dfrac{\dot{\mathbf{r}}}{|\dot{\mathbf{r}}|}$
- Speed of the trajectory: $\nu = |\dot{\mathbf{r}}|$





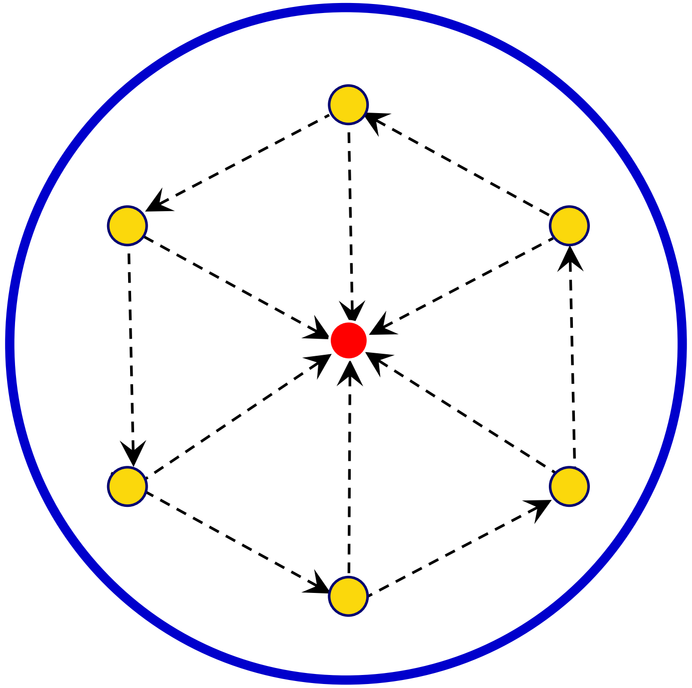
❑ Agent Dynamics:

$$\begin{aligned}\dot{\mathbf{r}}(t) &= \nu(t)\mathbf{x}(t)\\ \dot{\mathbf{x}}(t) &= \nu(t)u(t)\mathbf{y}(t)\\ \dot{\mathbf{y}}(t) &= -\nu(t)u(t)\mathbf{x}(t)\end{aligned}$$

❑ Beacon modeled as a particle with zero speed and arbitrary frame orientation.

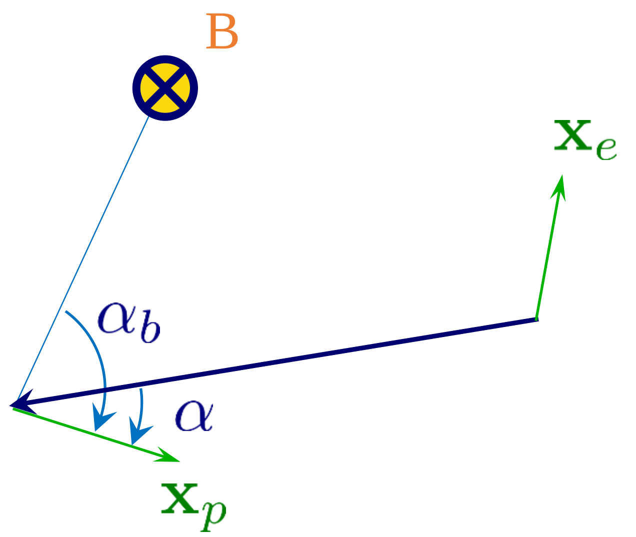
Modeling: Attention Graph

- Individual agents are perceived as vertices in a *directed graph*: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$
  - **Vertex Set**: $\mathcal{V} = \{1, 2, \cdots, n\}$
  - **Edge Set**: $\mathcal{E} = \left\{(i,j) \in \mathcal{V} \times \mathcal{V} \middle| \text{Agent } i \textit{ pays attention} \text{ to Agent } j\right\}$



Cycle with Spokes

Modeling: Beacon Referenced CB Pursuit Control Law



Weighting factor

- $\lambda \in [0, 1]$

$$u^i_{CB+B} = (1 - \lambda) u^i_{CB} - \lambda \mu^b_i \left(R(\alpha_b) \mathbf{y}_i \cdot \frac{\mathbf{r}_{i,b}}{|\mathbf{r}_{i,b}|} \right)$$

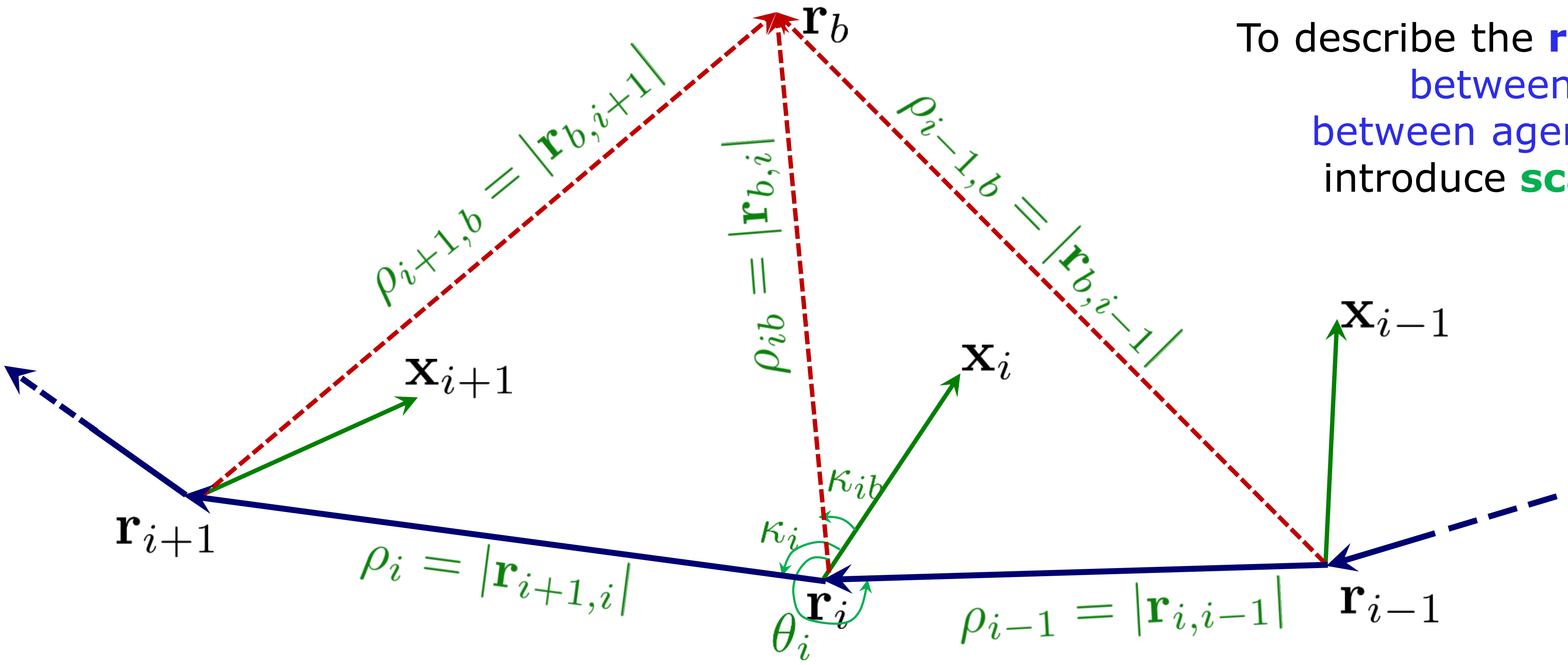
References only relative bearing to the beacon

Constant Bearing Pursuit Law:

$$u^i_{CB} = -\mu_i \left(R(\alpha_i) \mathbf{y}_i \cdot \frac{\mathbf{r}_{i,i+1}}{|\mathbf{r}_{i,i+1}|} \right) - \frac{1}{\nu_i |\mathbf{r}_{i,i+1}|} \left(\frac{\mathbf{r}_{i,i+1}}{|\mathbf{r}_{i,i+1}|} \cdot R(\pi/2) \dot{\mathbf{r}}_{i,i+1} \right)$$

$$R(\vartheta) = \begin{bmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{bmatrix} \in SO(2)$$

Reduction to Shape Space (of relative *Position* & *Orientation*)



To describe the **relative configuration** between individual agents, and between agents and the beacon, we introduce **scalar shape variables**.

- **Consistency Condition**

$$\rho_i \mathbb{I}_2 = \rho_{ib} R(\kappa_{ib} - \kappa_i) + \rho_{i+1,b} R(\kappa_{i+1,b} - \theta_{i+1})$$

- **Cycle Closure Constraint**

$$R\Big(\sum_{i=1}^{n} (\pi + \kappa_i - \theta_{i+1})\Big) = \mathbb{I}_2$$

Closed Loop Shape Dynamics

Assumptions

- Individual speeds are constant and equal. $(\nu_i = 1)$
- Controller gains are equal and common for every agent. $(\mu_i = \mu_{ib} = \mu)$
- The bearing angles for the neighbors are equal for every agent. $(\alpha_i = \alpha)$
- The bearing angles for the beacon are equal for every agent. $(\alpha_{ib} = \alpha_0)$

$$\dot{\rho}_i = -\big(\cos\kappa_i + \cos\theta_{i+1}\big)$$

$$\dot{\kappa}_i = -\mu\big[(1-\lambda)\sin(\kappa_i - \alpha) + \lambda\sin(\kappa_{ib} - \alpha_0)\big] + \frac{\lambda}{\rho_i}\big[\sin\kappa_i + \sin\theta_{i+1}\big]$$

$$\dot{\theta}_i = \dot{\kappa}_i - \frac{1}{\rho_i}\big[\sin\kappa_i + \sin\theta_{i+1}\big] + \frac{1}{\rho_{i-1}}\big[\sin\kappa_{i-1} + \sin\theta_i\big]$$

$$\dot{\rho}_{ib} = -\cos\kappa_{ib}$$

$$\dot{\kappa}_{ib} = \dot{\kappa}_i - \frac{1}{\rho_i}\big[\sin\kappa_i + \sin\theta_{i+1}\big] + \frac{1}{\rho_{ib}}\sin\kappa_{ib}$$

Subject to consistency and closure constraints.

Closed Loop Shape Dynamics

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- The bearing angles for the neighbors are equal for every agent. $(\alpha_i = \alpha)$
- The bearing angles for the beacon are equal for every agent. $(\alpha_{ib} = \alpha_0)$

$\dot{\rho}_i =$ Inter-agent range dynamics

$\dot{\kappa}_i =$ Line-of-sight Angle to Pursuee dynamics

$\dot{\theta}_i =$ Line-of-sight Angle to Pursuer dynamics

$\dot{\rho}_{ib} =$ Agent-Beacon range dynamics

$\dot{\kappa}_{ib} =$ Line-of-sight Angle to Beacon dynamics

Subject to consistency and closure constraints.

Relative Equilibrium (*BD,KSG,*ACC'15 - Proposition 4.1)

Proposition: Consider an n-agent beacon-referenced cyclic pursuit system, parametrized by μ, λ, α and α_0. Then, the only possible relative equilibria are **circling equilibria**. Furthermore, whenever $n\alpha$ is not an integral multiple of π, a circling equilibrium exists if there exists $m \in \mathbb{Z}$ such that

(i) $\lambda \cos \alpha_0 + (1-\lambda) \sin\left(\frac{m\pi}{n} - \alpha\right) > 0$, and

(ii) $\sin\left(\frac{m\pi}{n}\right) > 0$.

$$\kappa_i = \frac{m\pi}{n}, \qquad \theta_i = \left(1 - \frac{m}{n}\right)\pi$$

$$\kappa_{ib} = \pm\pi/2$$

$$\pm \begin{cases} + & \text{CCW} \\ - & \text{CW} \end{cases}$$

$$\rho_i = \frac{2\lambda \sin\left(\frac{m\pi}{n}\right)}{\mu(1-\lambda)\sin\left(\frac{m\pi}{n} - \alpha\right) \pm \mu\lambda\cos\alpha_0}, \qquad \rho_{ib} = \frac{\lambda}{\mu\lambda\cos\alpha_0 \pm \mu(1-\lambda)\sin\left(\frac{m\pi}{n} - \alpha\right)}$$

Relative Equilibrium: Local Stability Analysis

$$\zeta_i = \begin{bmatrix} \rho_i \\ \kappa_i \\ \theta_i \\ \rho_{ib} \\ \kappa_{ib} \end{bmatrix} - \begin{bmatrix} \rho_i^* \\ \kappa_i^* \\ \theta_i^* \\ \rho_{ib}^* \\ \kappa_{ib}^* \end{bmatrix}$$

Perturbation around an Equilibrium

$$\dot{\zeta}_i = A_0 \zeta_i + A_1 \zeta_{i+1} + A_{-1} \zeta_{i-1}$$

$$A_0 = \begin{bmatrix} 0 & * & 0 & 0 & 0 \\ * & * & 0 & 0 & * \\ * & * & * & 0 & * \\ 0 & 0 & 0 & 0 & * \\ * & * & 0 & * & * \end{bmatrix} \quad A_1 = \begin{bmatrix} 0 & 0 & * & 0 & 0 \\ 0 & 0 & * & 0 & 0 \\ 0 & 0 & * & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & * & 0 & 0 \end{bmatrix} \quad A_{-1} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Linearized Dynamics is Governed by:

$$\hat{A} = \begin{bmatrix} A_0 & A_1 & 0_5 & 0_5 & \cdots & A_{-1} \\ A_{-1} & A_0 & A_1 & 0_5 & \cdots & 0_5 \\ 0_5 & A_{-1} & A_0 & A_1 & \cdots & 0_5 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0_5 & 0_5 & 0_5 & 0_5 & \cdots & A_1 \\ A_1 & 0_5 & 0_5 & 0_5 & \cdots & A_0 \end{bmatrix}$$

Relative Equilibrium: Local Stability Analysis

Proposition: The eigenvalues of $\hat{A}$ are given by the union of the eigenvalues of

$$\begin{matrix} A_0 + A_1 + A_{-1} \\ A_0 + \omega A_1 + \omega^{-1} A_{-1} \\ \vdots \\ A_0 + \omega^{n-1} A_1 + \omega^{-(n-1)} A_{-1} \end{matrix}, \quad \text{where } \omega = e^{2\pi j/n} \text{ is the } n\text{-th root of unity.}$$

Theorem: A necessary condition for stability of a counter-clockwise (CCW) circling equilibrium is given by

- $\lambda \sin \alpha_0 + (1-\lambda) \cos \alpha^* > 0.$

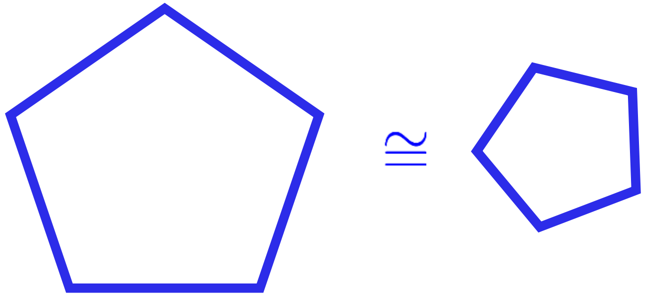
Furthermore, if n is even, the following conditions should also hold true:

- $\cos \alpha^* > 0,$
- $(1-\lambda)\left(\cos \alpha^* + a \cot(\frac{m\pi}{n})\right) + \lambda \sin \alpha_0 > 0,$
- $(1-\lambda)\left(a^2 + \left(b + a(1-\lambda)\cot(\frac{m\pi}{n})\right)\cos \alpha^*\right)\cot(\frac{m\pi}{n}) + \lambda a \sin \alpha_0 > 0.$

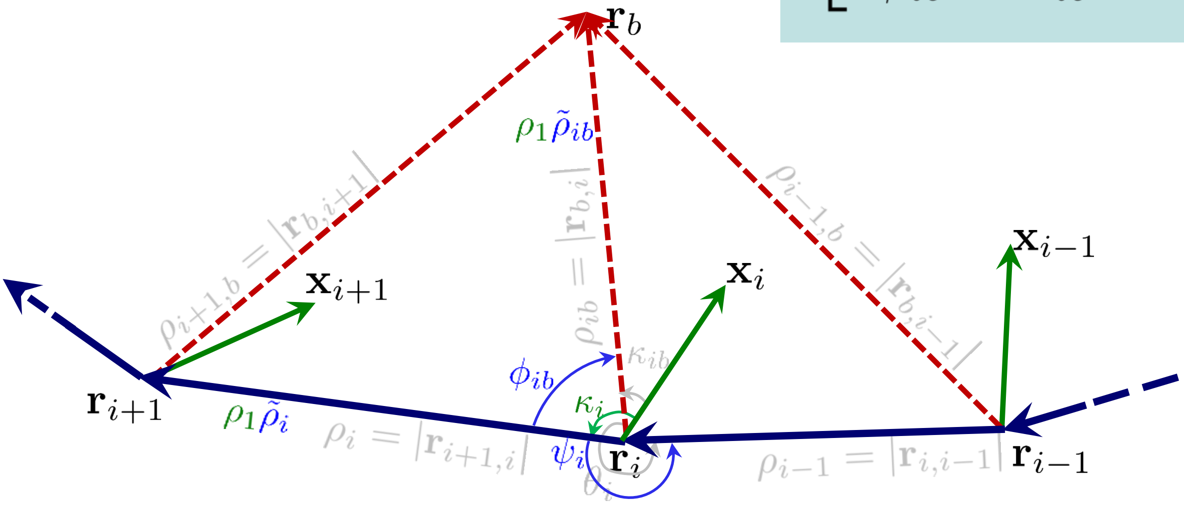
$a = \cos \alpha_0 + (\frac{1}{\lambda} - 1)\sin \alpha^*, \qquad \alpha^* = \left(\frac{m\pi}{n}\right) - \alpha$

Extraction of Pure Shape

- Change of variables allow us to separate out the SCALING aspect of the formation.



$$\begin{bmatrix} \rho_i \\ \kappa_i \\ \theta_i \\ \rho_{ib} \\ \kappa_{ib} \end{bmatrix}_{i=\{1,\dots,n\}} \Rightarrow \begin{bmatrix} \rho_1 \\ \kappa_1 \end{bmatrix}, \quad \begin{bmatrix} \tilde{\rho}_i \triangleq \frac{\rho_i}{\rho_1} \\ \tilde{\rho}_{ib} \triangleq \frac{\rho_{ib}}{\rho_1} \\ \tilde{\kappa}_i \triangleq \kappa_i - \kappa_{i+1} \\ \psi_i \triangleq \theta_i - \kappa_i \\ \phi_{ib} \triangleq \kappa_{ib} - \kappa_i \end{bmatrix}_{i=\{1,\dots,n\}}$$



- Constraints:

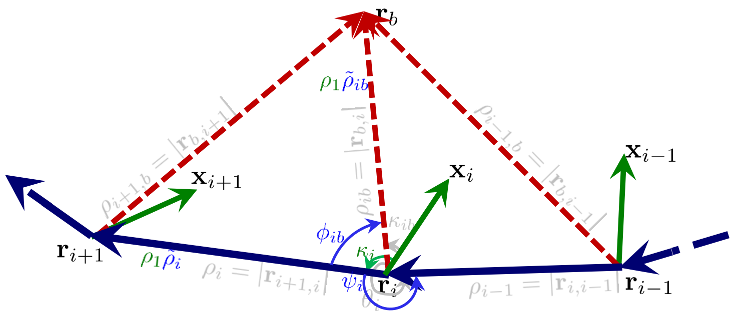
$$R\left(\sum_{i=1}^{n}(\pi - \psi_i)\right) = \mathbb{I}_2$$

$$\tilde{\rho}_{ib}R(\phi_{ib}) + \tilde{\rho}_{i+1,b}R(\phi_{i+1,b} - \psi_{i+1}) = \tilde{\rho}_i\mathbb{I}_2$$

$$\sum_{i=1}^{n}\tilde{\kappa}_i = 0$$

$$\tilde{\rho}_1 = 1$$

Dynamics of the new variables



$$\dot{\kappa}_1 = -\mu\big[(1-\lambda)\sin(\kappa_1-\alpha) + \lambda\sin(\phi_{1b}+\kappa_1-\alpha_0)\big] + \frac{2\lambda\sin\left(\frac{\Phi_1}{2}\right)\cos\left(\frac{\Psi_1}{2}\right)}{\rho_1}$$

$$\dot{\rho}_1 = -2\cos\left(\frac{\Phi_1}{2}\right)\cos\left(\frac{\Psi_1}{2}\right)$$

$$\dot{\tilde{\kappa}}_i = -2\mu\left[(1-\lambda)\sin\left(\frac{\tilde{\kappa}_i}{2}\right)\cos\left(\frac{\kappa_i^+-2\alpha}{2}\right) + \lambda\sin\left(\frac{\phi_{ib}-\phi_{i+1,b}+\tilde{\kappa}_i}{2}\right)\cos\left(\frac{\phi_{ib}+\phi_{i+1,b}+\kappa_i^+-2\alpha_0}{2}\right)\right]$$
$$+\frac{2\lambda}{\rho_1}\left[\frac{\sin\left(\frac{\Phi_i}{2}\right)\cos\left(\frac{\Psi_i}{2}\right)}{\tilde{\rho}_i} - \frac{\sin\left(\frac{\Phi_{i+1}}{2}\right)\cos\left(\frac{\Psi_{i+1}}{2}\right)}{\tilde{\rho}_{i+1}}\right]$$

$$\dot{\tilde{\rho}}_i = \frac{2}{\rho_1}\left[\tilde{\rho}_i\cos\left(\frac{\Phi_1}{2}\right)\cos\left(\frac{\Psi_1}{2}\right) - \cos\left(\frac{\Phi_i}{2}\right)\cos\left(\frac{\Psi_i}{2}\right)\right]$$

$$\dot{\psi}_i = \frac{2}{\rho_1}\left[\frac{\sin\left(\frac{\Phi_{i-1}}{2}\right)\cos\left(\frac{\Psi_{i-1}}{2}\right)}{\tilde{\rho}_{i-1}} - \frac{\sin\left(\frac{\Phi_i}{2}\right)\cos\left(\frac{\Psi_i}{2}\right)}{\tilde{\rho}_i}\right]$$

$$\dot{\tilde{\rho}}_{ib} = \frac{1}{\rho_1}\left[2\tilde{\rho}_{ib}\cos\left(\frac{\Phi_1}{2}\right)\cos\left(\frac{\Psi_1}{2}\right) - \cos\left(\phi_{ib}+\kappa_1+\sum_{j=i}^{n}\tilde{\kappa}_j\right)\right]$$

$$\dot{\phi}_{ib} = \frac{1}{\rho_1}\left[\frac{1}{\tilde{\rho}_{ib}}\sin\left(\phi_{ib}+\kappa_1+\sum_{j=i}^{n}\tilde{\kappa}_j\right) - \frac{2\sin\left(\frac{\Phi_i}{2}\right)\cos\left(\frac{\Psi_i}{2}\right)}{\tilde{\rho}_i}\right]$$

$$\Phi_i \triangleq 2\kappa_1 + \tilde{\kappa}_i + \psi_{i+1} + 2\sum_{j=i+1}^{n}\tilde{\kappa}_j$$

$$\Psi_i \triangleq \tilde{\kappa}_i - \psi_{i+1}$$

Pure Shape Equilibria

$$\cos\left(\frac{\Phi_i}{2}\right) \neq 0, \; \cos\left(\frac{\Psi_i}{2}\right) \neq 0, \; \sin\left(\frac{\Phi_i}{2}\right) \neq 0$$

$\dot{\tilde{\rho}}_i = 0$, $\dot{\psi}_i = 0$ →

$$\Phi_1 = \Phi_2 = \cdots = \Phi_n = \Phi$$

$\dot{\tilde{\rho}}_{ib} = 0$, $\dot{\phi}_{ib} = 0$ →

$$\sin\left(\phi_{ib} - \phi_{i-1,b} - \tilde{\kappa}_{i-1}\right) = 0$$

→ $= 2k\pi$

→ $= (2k+1)\pi$

Pure Shape Equilibria

$$\cos\left(\frac{\Phi_i}{2}\right) \neq 0,\ \cos\left(\frac{\Psi_i}{2}\right) \neq 0,\ \sin\left(\frac{\Phi_i}{2}\right) \neq 0$$

$\dot{\tilde{\rho}}_i = 0$, $\dot{\psi}_i = 0$ →

$$\Phi_1 = \Phi_2 = \cdots = \Phi_n = \Phi$$

$\dot{\tilde{\rho}}_{ib} = 0$, $\dot{\phi}_{ib} = 0$ →

$$\sin\left(\phi_{ib} - \phi_{i-1,b} - \tilde{\kappa}_{i-1}\right) = 0$$

$= 2k\pi$ or $= (2k+1)\pi$

$\dot{\tilde{\kappa}}_i = 0$ →

$$-2\mu(1-\lambda)\sin\left(\frac{\tilde{\kappa}_i}{2}\right)\cos\left(\frac{\tilde{\kappa}_i}{2} + \kappa_1 - \alpha + \sum_{j=i+1}^{n} \tilde{\kappa}_j\right) = 0$$

Pure Shape Equilibria

$$\cos\left(\frac{\Phi_i}{2}\right) \neq 0, \ \cos\left(\frac{\Psi_i}{2}\right) \neq 0, \ \sin\left(\frac{\Phi_i}{2}\right) \neq 0$$

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$$= 2k\pi$$

$$= (2k+1)\pi$$

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$$-2\mu(1-\lambda)\sin\left(\frac{\tilde{\kappa}_i}{2}\right)\cos\left(\frac{\tilde{\kappa}_i}{2} + \kappa_1 - \alpha + \sum_{j=i+1}^{n} \tilde{\kappa}_j\right) = 0$$

Pure Shape Equilibria

Proposition: For any $k \in \{1, 2, \ldots, n\}$, the manifold $\mathcal{M}_k$ defined by

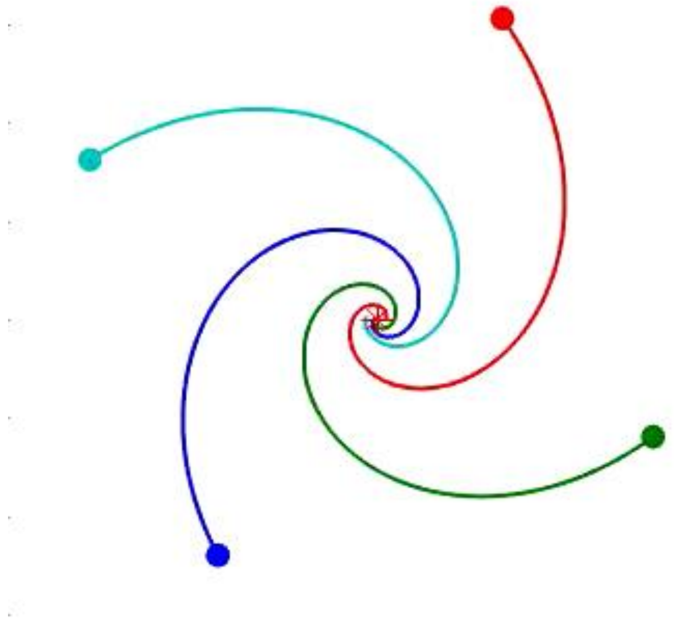
$$\mathcal{M}_k \triangleq \Bigg\{ \kappa_1, \rho_1, \{\tilde{\kappa}_i, \psi_i, \phi_{ib}, \tilde{\rho}_i, \tilde{\rho}_{ib}\}_{i=1}^n \;\Bigg|\; \tilde{\kappa}_i = 0, \tilde{\rho}_i = 1, \psi_i = \left(\frac{n-2k}{n}\right)\pi,\ \phi_{ib} = \left(\frac{n-2k}{2n}\right)\pi,\ \tilde{\rho}_{ib} = \frac{1}{2\sin\left(\frac{k\pi}{n}\right)} \Bigg\},$$

is nonempty and invariant under the closed loop dynamics. Furthermore, the 2-dimensional reduced dynamics on the invariant manifold is given by

$$\dot{\kappa}_1 = -\mu\left[(1-\lambda)\sin(\kappa_1 - \alpha) + \lambda\cos\left(\kappa_1 - \frac{k\pi}{n} - \alpha_0\right)\right] + \frac{2\lambda}{\rho_1}\cos\left(\kappa_1 - \frac{k\pi}{n}\right)\sin\left(\frac{k\pi}{n}\right)$$

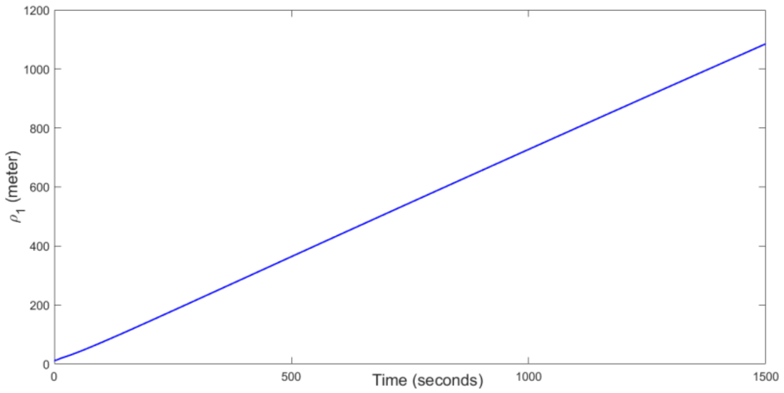
$$\dot{\rho}_1 = -\cos\kappa_1 + \cos\left(\kappa_1 - \frac{2k\pi}{n}\right).$$

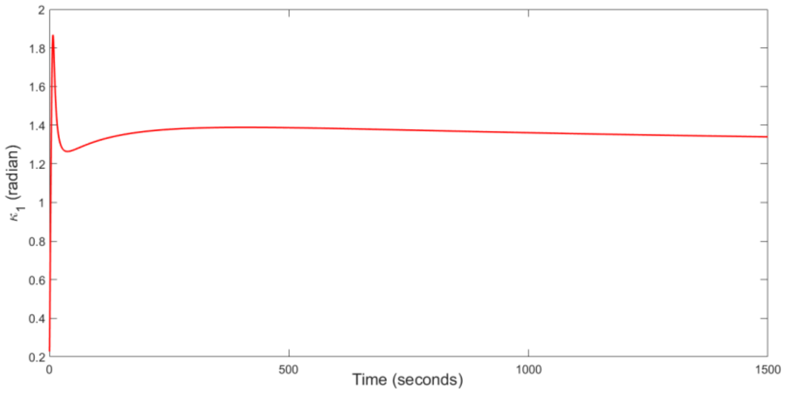
Numerical Simulation

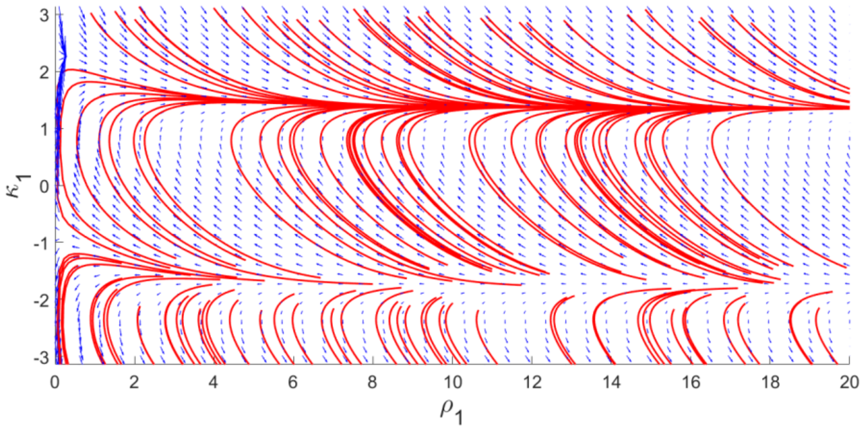


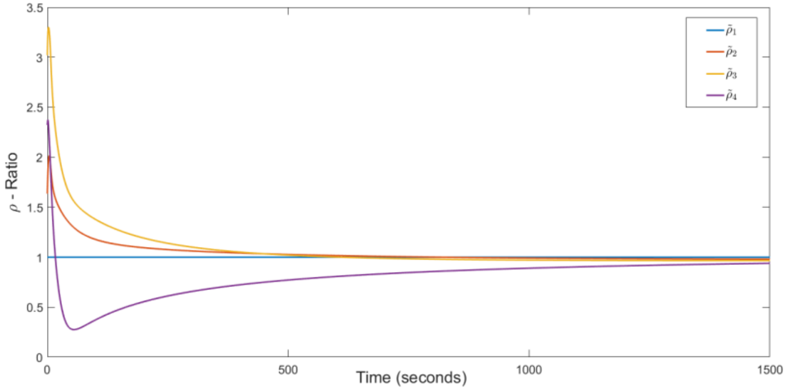
Simulation Parameters:

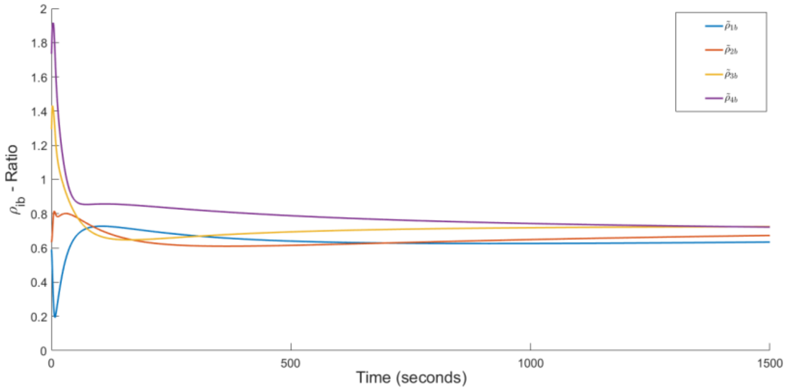
$$\alpha = \frac{\pi}{4}, \quad \alpha_0 = \frac{5\pi}{6}, \quad \lambda = 0.5, \quad \mu = 1$$





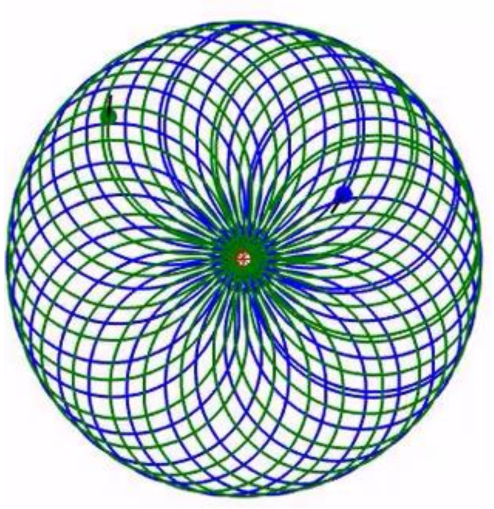




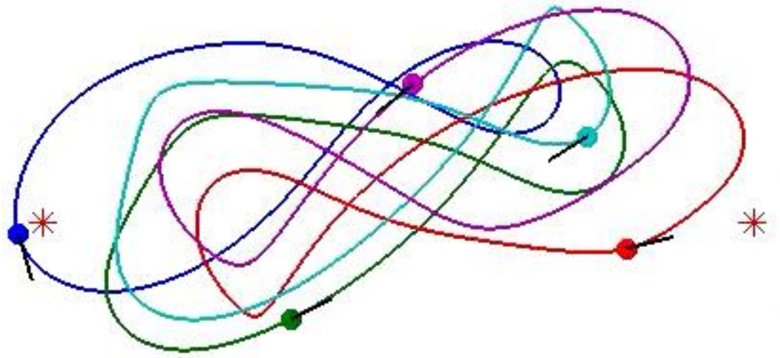


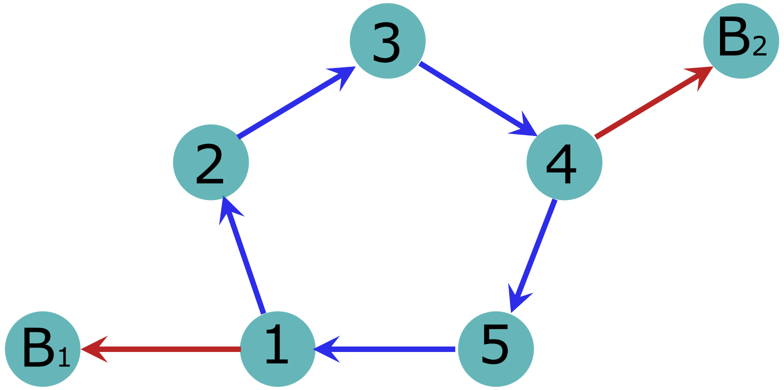
Future Work

- Coverage through Beacon-referenced Cyclic Pursuit



- Multiple Beacon





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  - USNA

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Thank You!