Controllability in a Network of Linear Dynamical Systems

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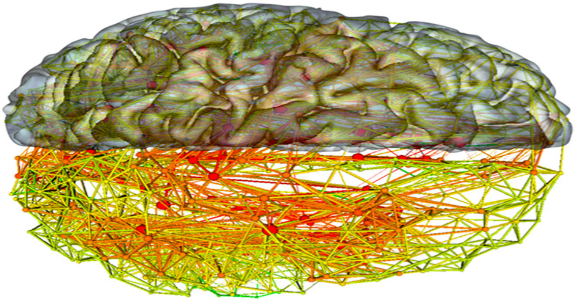
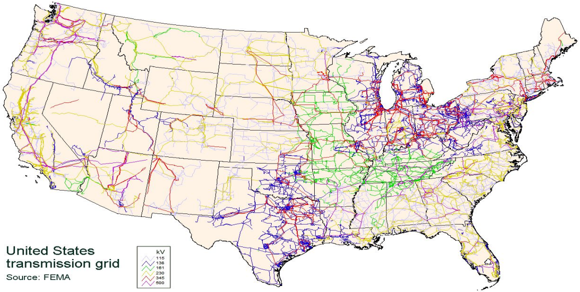
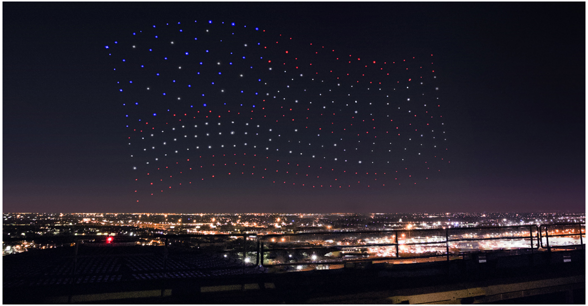
Controllability - A Key Property Across Multiple Domains







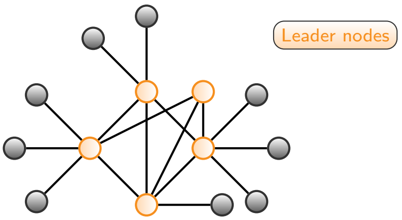


FIGURE: Diffusively coupled networked multi-agent system

- Number of agents in the network: N
- State of agent-i: $x_i \in \mathbb{R}^n$
- $v_i \in \mathbb{R}^p$ and $y_i \in \mathbb{R}^p$ denote the input and output of agent-i
- Dynamics of agent-i:

$$\begin{aligned} \dot{x}_i &= Ax_i + Bv_i \\ y_i &= Cx_i \end{aligned} \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times p}$ and $C \in \mathbb{R}^{p\times n}$

- Agents are interconnected via diffusive coupling over an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \Gamma)$ with Laplacian $L \triangleq D - \Gamma$
- Leader set: $S_m \triangleq \{s_1, \ldots, s_m\} \subseteq \mathcal{V}$
- Control input to individual agent: $$v_i = \begin{cases} u_i + \sum_{j=1}^{N} \gamma_{ij} C(x_j - x_i) & i \in S_m \\ \sum_{j=1}^{N} \gamma_{ij} C(x_j - x_i) & \text{otherwise} \end{cases} \tag{2}$$

Controllability

Consider the networked multi-agent system defined by (1)-(2). This system is **controllable** if for any $\xi_\flat, \xi_\sharp \in \mathbb{R}^{Nn}$, there exist a finite $\tau > 0$ and control input $u_i, i \in S_m$ such that the control input transfers overall state of the system (1)-(2) from $[x_1(0), \ldots, x_N(0)] = \xi_\flat$ to $[x_1(\tau), \ldots, x_N(\tau)] = \xi_\sharp$.

- **Define**: $X = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} \in \mathbb{R}^{Nn}$, $\quad U = \begin{bmatrix} u_{s_1} \\ \vdots \\ u_{s_m} \end{bmatrix} \in \mathbb{R}^{mp}$, $\quad G = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 1 \text{ (row - } s_1) & \cdots & 1 \text{ (row - } s_m) \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}$
- **Overall dynamics of the network:**

$$\dot{X} = (I_N \otimes A - L \otimes BC)X + (G \otimes B)U \tag{3}$$

- Spectral decomposition of L:

$$L = \Psi \Lambda \Psi^{-1} = \begin{bmatrix} \psi_{1,1} & \psi_{1,2} & \cdots & \psi_{1,N} \\ \psi_{2,1} & \psi_{2,2} & \cdots & \psi_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ \psi_{N,1} & \psi_{N,2} & \cdots & \psi_{N,N} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_N \end{bmatrix} \begin{bmatrix} \psi_{1,1} & \psi_{2,1} & \cdots & \psi_{N,1} \\ \psi_{1,2} & \psi_{2,2} & \cdots & \psi_{N,2} \\ \vdots & \vdots & \ddots & \vdots \\ \psi_{1,N} & \psi_{2,N} & \cdots & \psi_{N,N} \end{bmatrix}$$

Theorem 1

Consider the networked multi-agent system defined by (3). This system is controllable if and only if the following conditions hold true:

(I) For each $\lambda_i \in \mathrm{spec}(L)$, none of the left eigenvectors of $(A - \lambda_i BC)$ are orthogonal to B.

(II) $\sum_{j=1}^{m} \psi^2_{s_j,i} \neq \mathbf{0}$ for all $i = 1, \ldots, N$, i.e., the pair $[L, G]$ is controllable.

- **Controllability Gramian**: $W_c(T, S_m) = \int\limits_0^T e^{(I_N \otimes A - L \otimes BC)\sigma} (GG^\intercal \otimes BB^\intercal) e^{(I_N \otimes A - L \otimes BC)^\intercal \sigma} d\sigma$
- Trace of the controllability Gramian has an inverse relationship with the lower bound on control effort, and therefore its maximization can be used for optimal leader selection[1].

Proposition 1
Trace of the controllability Gramian $W_c(T, S_m)$ is a modular set function, and hence the set of $(m+1)$ optimal leaders contains the set of m optimal leaders.

THANK YOU!

[1]Summers T. H., Lygeros J. (2014) "Optimal Sensor and Actuator Placement in Complex Dynamical Networks", *Proceedings of the 19th IFAC World Congress*, 47(3):3784-3789.