Heterogeneity and Synchronization of Coupled Neuronal Networks

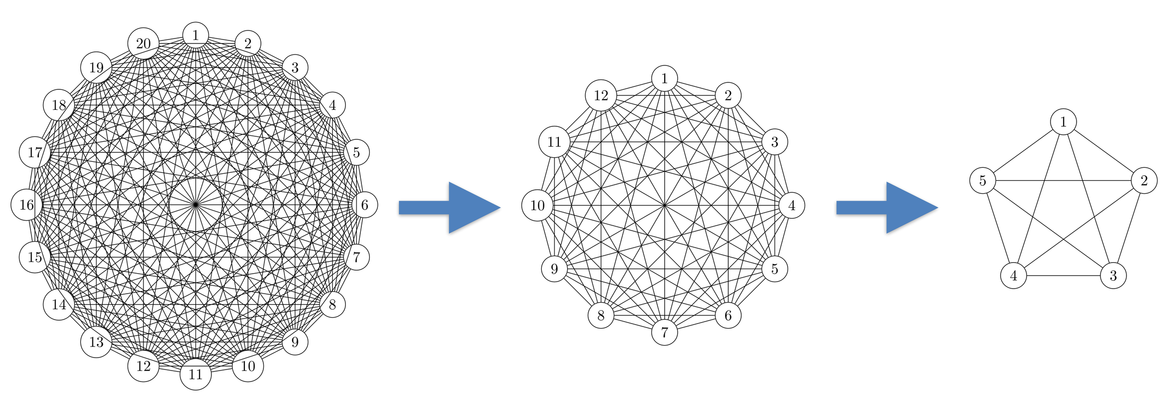
Elizabeth Davison, Biswadip Dey, and Naomi Leonard

PRINCETON
School of Engineering and Applied Science

Motivation and Problem Statement

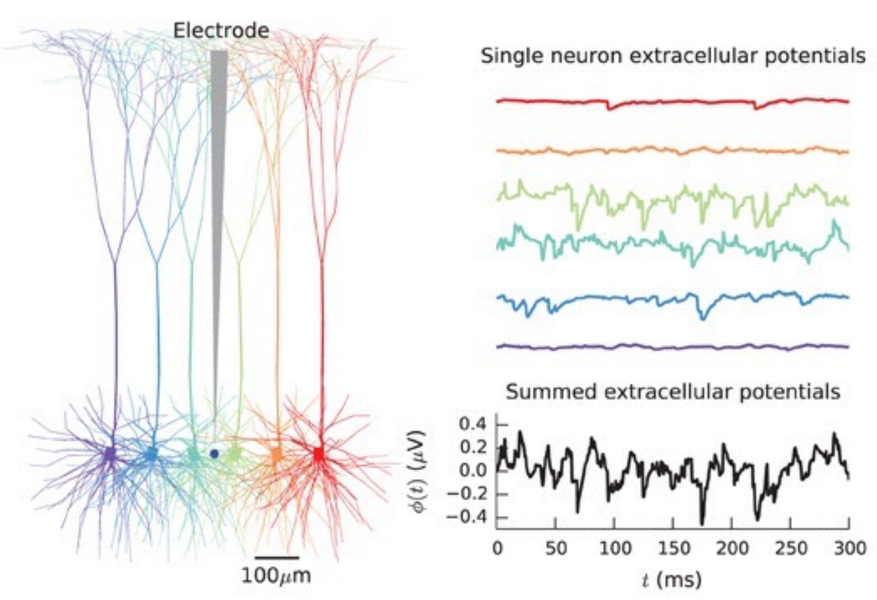
Synchronization in networks of FitzHugh-Nagumo (FN) oscillators coupled via gap junctions

- *Semi-passivity* guarantees bounded solutions
- *Sufficient condition* for complete synchronization
- Removal of constraints on *graph structure*
- Variation in *external inputs*



Studying the dynamics of an ensemble of model neurons gives us insight into the behavior of biological neuronal networks

- Connections between neuronal spiking and mesoscale properties, e.g. LFP. [1]
- Function in certain brain systems (e.g. basal ganglia) are closely related to synchronization; external influence such as DBS can alter synchronization properties
  - Desynchronization especially applicable in clinical cases
- Preservation of graph characteristics across length scales



FitzHugh-Nagumo Equations

- Reduced model of action potential for the *i*-th neuron
- y_i excitatory (fast)
- z_i inhibitory (slow)
- Linear electrical coupling u_i

$$\dot{y}_i = y_i - \frac{y_i^3}{3} - z_i + I_i + u_i$$

$$\dot{z}_i = \phi(y_i + a - bz_i),$$

$$u_i = \gamma \sum_j (y_j - y_i).$$

A system is strictly semi-passive in a region D if there exists a function V such that $\dot{V} \leq y^\top u - H(x)$, where $H(x) > 0$ outside a ball of radius ρ.[2]

Sufficient Conditions for Synchronization

Semi-passivity gives us bounded solutions $\quad |y| \leq (3I)^{1/3} \quad |z| \leq \frac{a}{b}$

Choose Lyapunov function

$$V = \frac{1}{2}\sum_{i=1}^{n}\left[(y_1 - y_i)^2 + (z_1 - z_i)^2\right]$$

For sufficiently large coupling, the completely synchronized state is stable.

$$\gamma > \frac{(\phi-1)^2}{4kb\phi} + \frac{1}{k} + \frac{(3I)^{2/3}}{3k}$$

Rewrite states as average and k-1 differences from average

$$\dot{\xi}_1 = \sum_{j=1}^{k}\frac{\dot{y}_j}{k} = \xi_1 - \frac{1}{3k}\sum_{j=1}^{k} y_j^3 - \zeta_1 + I$$

$$\dot{\xi}_i = y_i - \sum_{j=1}^{k}\frac{y_j}{k} = \xi_i - \frac{y_i^3}{3} + \frac{1}{k}\sum_{j=1}^{k}\frac{y_j^3}{3} - \zeta_i - k\gamma\xi_i$$

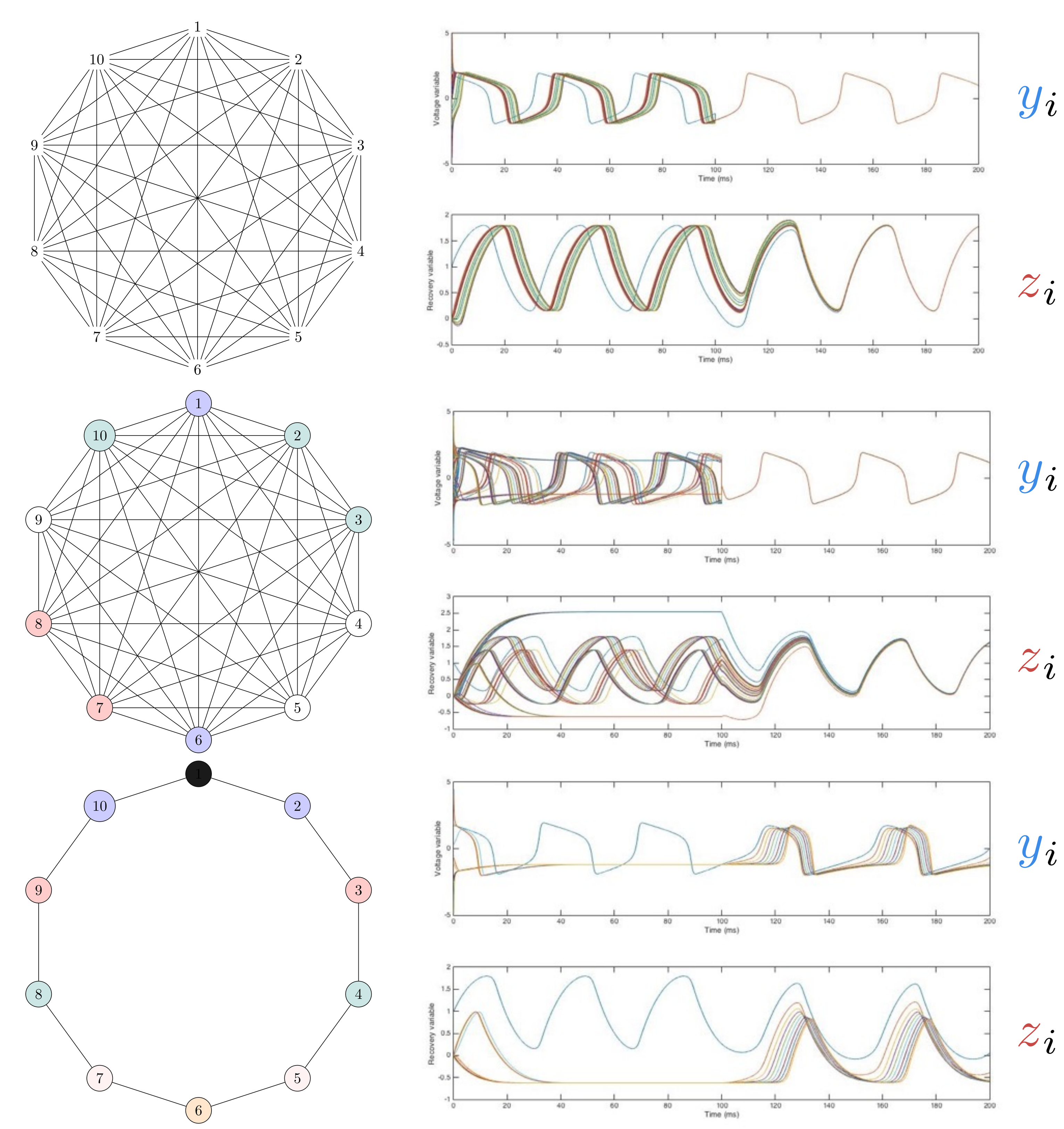
Transformed dynamics:

$$\dot{\zeta}_1 = \sum_{j=1}^{k}\frac{\dot{z}_j}{k} = \phi(\xi_1 + a - b\zeta_1)$$

$$\dot{\zeta}_i = z_i - \sum_{j=1}^{k}\frac{z_j}{k} = \phi(\xi_i - b\zeta_i)$$

A singular perturbation for $k\gamma \gg 1$ results in a single FN oscillator

Synchronized States



Further Considerations: Graph Structure and External Inputs

Arbitrary graph:

- Given sufficient condition on second smallest eigenvalue of the coupling matrix

$$\Gamma = \begin{pmatrix} \sum\gamma_{1j} & -\gamma_{12} & \cdots & -\gamma_{1k} \\ -\gamma_{21} & \sum\gamma_{2j} & \ddots & -\gamma_{2k} \\ \vdots & \ddots & \ddots & \vdots \\ -\gamma_{k1} & -\gamma_{k2} & \cdots & \sum\gamma_{kj} \end{pmatrix} \qquad \mathbf{u} = -\Gamma\mathbf{y}$$

- Stable complete synchronization state when

$$\lambda_2 > \frac{1}{4b\phi} + 1 + \frac{(3I)^{2/3}}{3}$$

Differing inputs:

- Cluster synchronization arises in heterogeneous systems.[3]
- Stability argument is identical to earlier cases: can use a superposition of Lyapunov functions

$$V = \sum_{i=1}^{n} V_i$$

- Complete graph: clusters form based directly on inputs
  - We find a sufficient bound for independent synchronization of all clusters

$$\gamma > \frac{(\phi-1)^2}{4kb\phi} + \frac{1}{k} + \frac{(3I_{\max})^{2/3}}{3k},$$
$$\text{where } I_{\max} = \max_{i\in[1,k]} I_i.$$

- Other graph structures - nontrivial to determine clusters
- High coupling limit - again approaches a single FN oscillator

Conclusions

- We find an explicit bound on coupling that guarantees complete synchronization in networks of homogeneous FN oscillators
  - Singular perturbation reduces to single FN oscillator
- In external input case, we cannot get complete synchronization
  - New synchronization state is possible: cluster synchronization
  - Proved stability of cluster synchronization in complete graph when coupling is above a threshold

References:

[1] H. Lindén et. al. *LFPy: a tool for biophysical simulation of extracellular potentials generated by detailed model neurons*. Frontiers in Neuroscience 7 (2014).

[2] E. Steur, I. Tyukin, and H. Nijmeijer. *Semi-passivity and synchronization of diffusively coupled neuronal oscillators*. Physica D: Nonlinear Phenomena 238.21, (2009).

[3] F. Sorrentino and E. Ott. *Network synchronization of groups*. Phys Rev E. 76, (2007).