Synchronization and Related Phenomena in Networks of Diffusively-Coupled Fitzhugh-Nagumo Oscillators

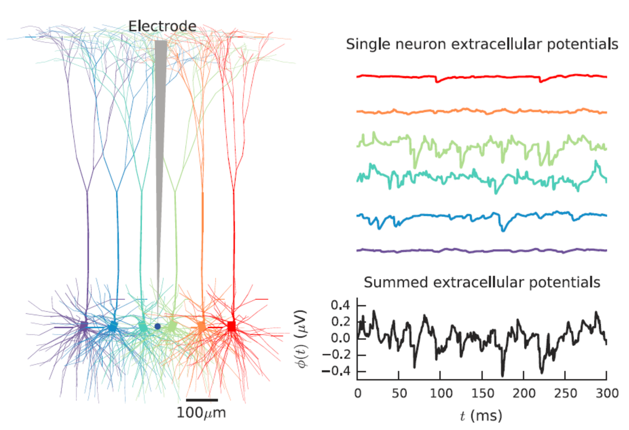
PRINCETON UNIVERSITY

Biswadip Dey

(*joint work with Elizabeth N. Davison, Zahra Aminzare and Naomi Ehrich Leonard*)

Motivation and Background

Smithsonian Magazine (2016)



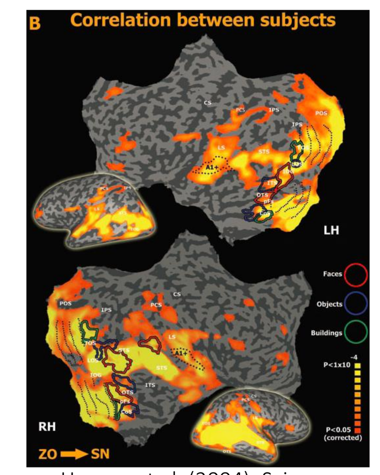
Linden et al. (2014). *Frontiers in Neuroinformatics*

- Synchronized activity is crucial for brain function:
  - Occurs at multiple levels (basal ganglia, local field potential)
  - Related to many pathological conditions (epilepsy)
- Insight about synchronization can lead to advances in:
  - Deep Brain Stimulation, Transcranial Stimulation
  - System Identification
  - Testable predictions
  - Measurable efficacy metrics for disease treatment

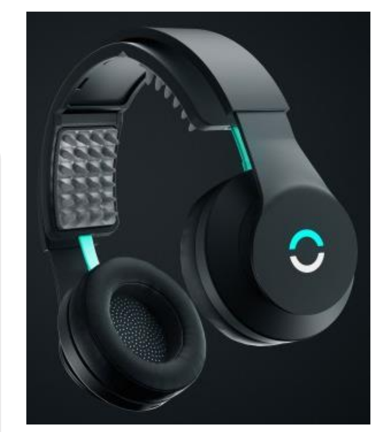
For a network of N-oscillators with state $\mathbf{x}_i \in \mathbb{R}^n$, $1 \le i \le N$ we define the synchronization manifold as

$$\mathcal{S} = \{\mathbf{x}_1 = \mathbf{x}_2 = \cdots = \mathbf{x}_N | \, X^i \in \mathbb{R}^n\}.$$

Then the network synchronizes completely if the state trajectories converge to $\mathcal{S}$ in some appropriate norm.



Hasson et al. (2004). Science



MIT Tech. Review (2016)

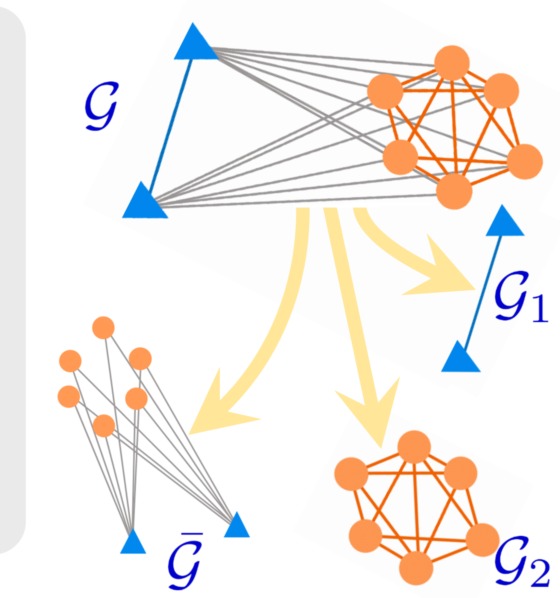
Cluster synchronization in a network of non-identical oscillators

For a network of N-oscillators with state $\mathbf{x}_i \in \mathbb{R}^n, 1 \le i \le N$ we define the cluster synchronization manifold as

$$\mathcal{S}^K = \{\mathbf{x}_1 = \cdots = \mathbf{x}_{c_1}, \; \ldots, \; \mathbf{x}_{N-c_K+1} = \cdots = \mathbf{x}_N | \, \mathbf{x}_i \in \mathbb{R}^n\},$$

where $1 \le K \le N$, and there exists $1 \le c_1, \ldots, c_K \le N$ such that $c_1 + \cdots + c_K = N$. Then the network synchronizes in clusters if the state trajectories converge to $\mathcal{S}^K$ in some appropriate norm. The k-th cluster is defined as

$$\mathcal{C}^k = \left\{ \sum_{l=1}^{k-1} c_l + 1, \; \sum_{l=1}^{k-1} c_l + 2, \; \cdots, \; \sum_{l=1}^{k} c_l \right\}.$$



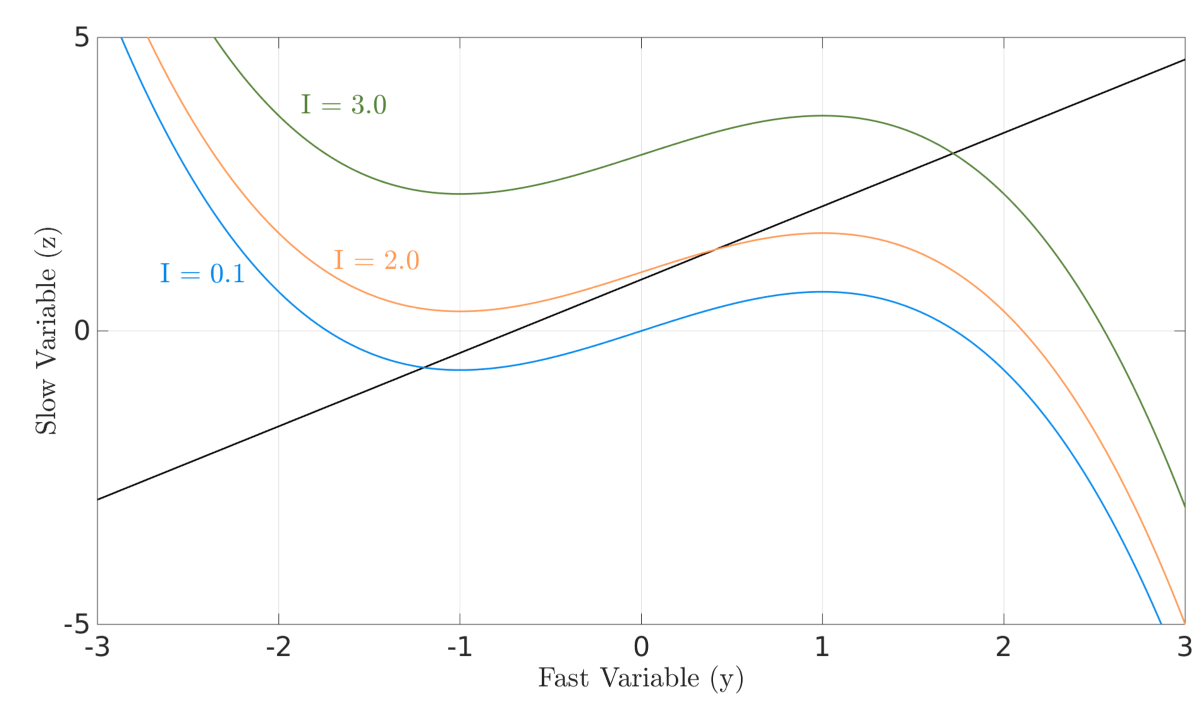
- Oscillators belonging to the same cluster (*e.g.* $\mathcal{C}^k$) are identical.
- **Cluster-input-Equivalence Condition** (k-th cluster): $\sum_{m \in \mathcal{C}^l} \gamma_{im} = \sum_{m \in \mathcal{C}^l} \gamma_{jm}, \quad \forall l \in \{1, \cdots, K\} \setminus k, \; \forall i, j \in \mathcal{C}^k$
- **Sufficient Condition:** $\gamma > \dfrac{1 + \alpha_k}{\lambda_2(\mathcal{G}_k) + \lambda_2(\bar{\mathcal{G}})}, \quad \forall 1 \le k \le K, \; \alpha_k = \dfrac{\left(\epsilon^{(k)} p - 1/p\right)^2}{4 b^{(k)} \epsilon^{(k)}}, \; p = \max_{1 \le l \le K} \dfrac{1}{\sqrt{\epsilon^{(l)}}}$

Fitzhugh-Nagumo (FN) neuronal oscillator

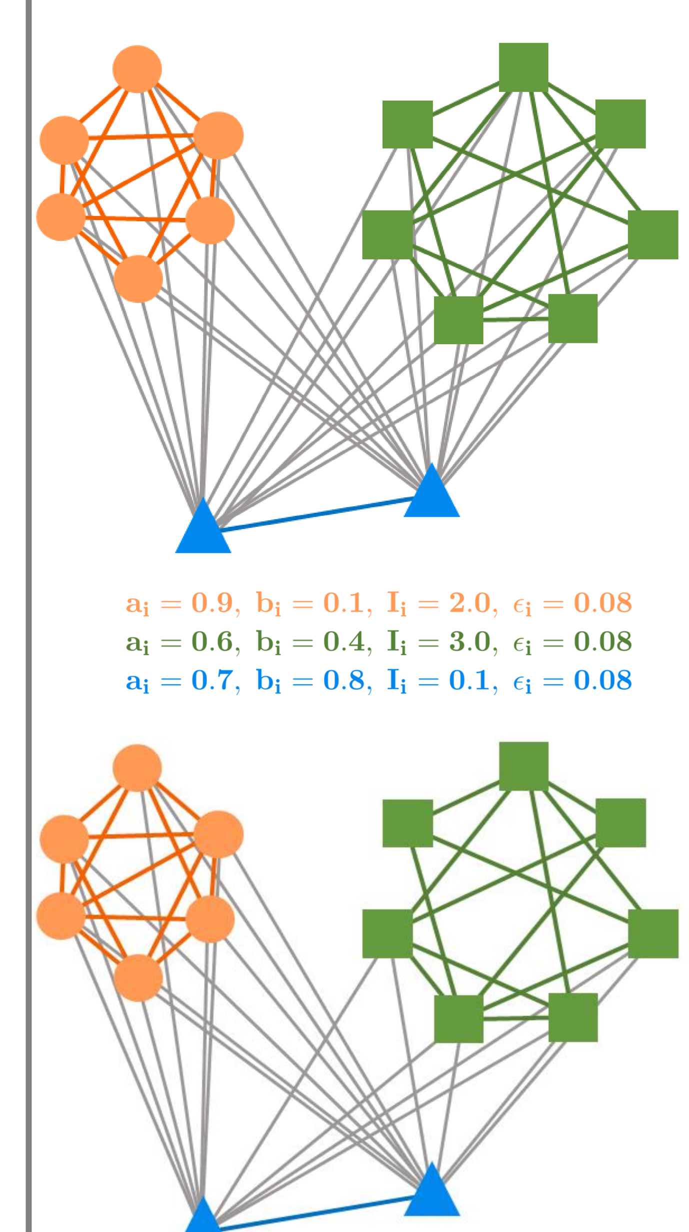
- Second-order Dynamics (Relaxation Oscillator):
  - Fast Dynamics: $\dot{y} = y - \dfrac{y^3}{3} - z + I$ (Membrane Potential)
  - Slow Dynamics: $\dot{z} = \epsilon(y - bz + a)$ (Recovery Variable)

  I → External Input; ϵ → Time Scale Separation

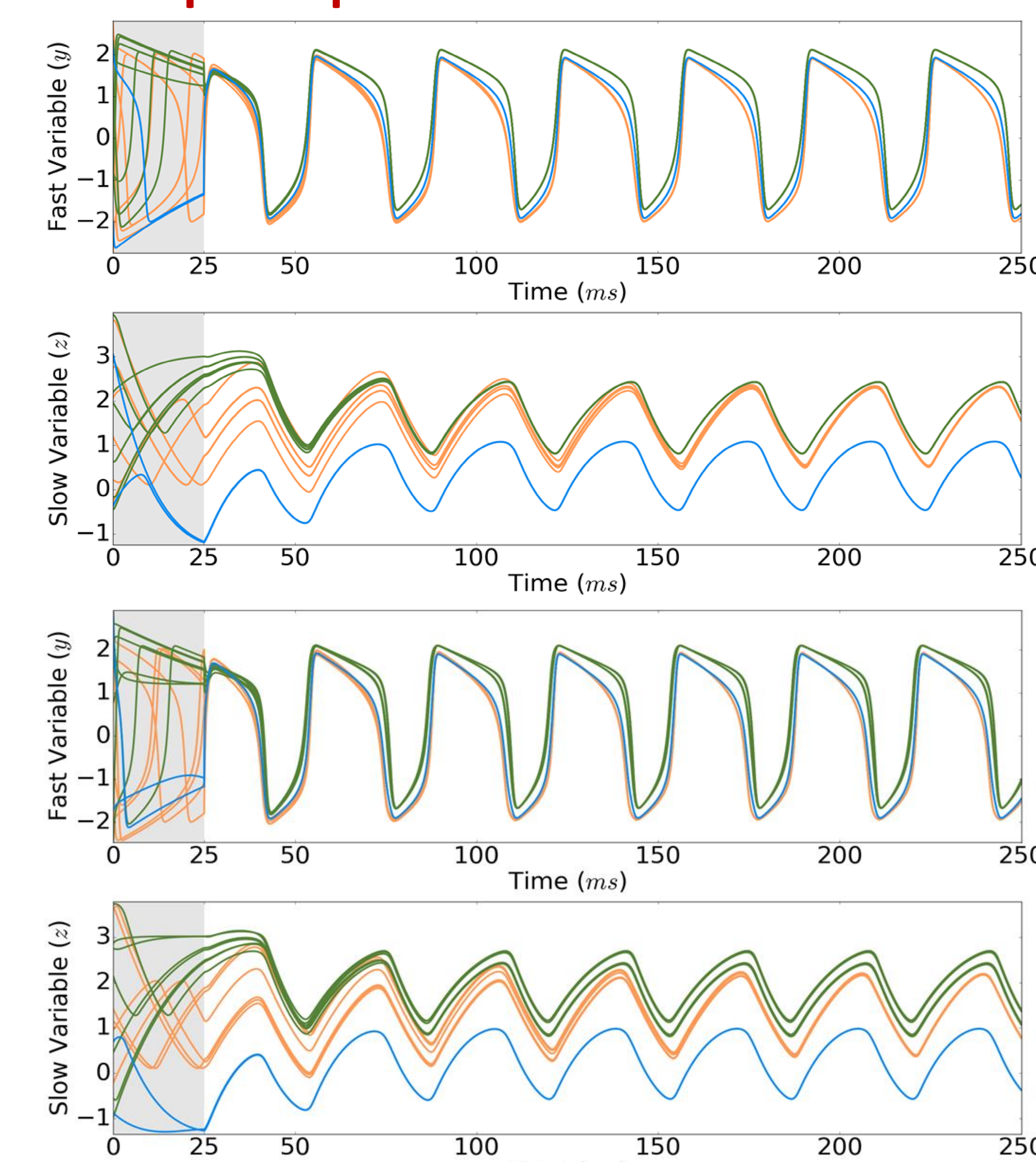
- The dynamics of a FN Oscillator are strictly semi-passive, i.e. outside a ball around the origin, it behaves as a strictly passive system.



Necessity of cluster-input-equivalence condition



$a_i = 0.9, \; b_i = 0.1, \; I_i = 2.0, \; \epsilon_i = 0.08$
$a_i = 0.6, \; b_i = 0.4, \; I_i = 3.0, \; \epsilon_i = 0.08$
$a_i = 0.7, \; b_i = 0.8, \; I_i = 0.1, \; \epsilon_i = 0.08$



Diffusively-coupled network of FN oscillators

- Individual dynamics:

$$\dot{y}_i = y_i - \frac{y_i^3}{3} - z_i + I_i + u_i$$
$$\dot{z}_i = \epsilon_i (y_i - b_i z + a_i)$$

u_i → External Input

- Electrical gap junction coupling:

$$u_i = \gamma \sum_{j=1}^{n} \gamma_{ij} (y_j - y_i)$$

γ_{ij} → Edge weights of the network graph; γ → Coupling Strength

- We assume the network graph ($\mathcal{G}$) to be connected, weighted and undirected.
- The closed-loop system has ultimately bounded solutions.

Synchronization in networks of identical oscillators

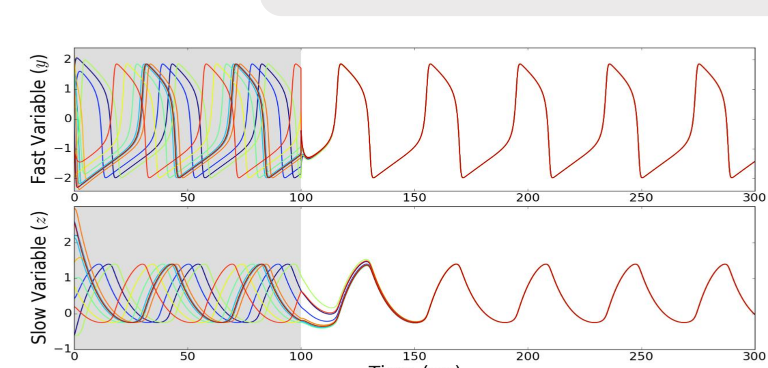
Model Parameters: $a_i = a, \; b_i = b, \; \epsilon_i = \epsilon, \; I_i = I \quad \forall i$

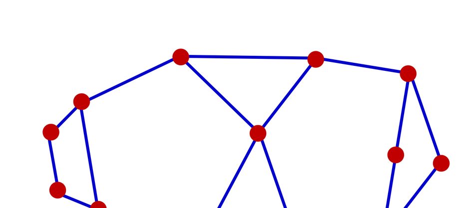
- Non-Smooth Lyapunov Analysis:[1]
  - Sufficient condition in terms of a lower bound on the second-smallest eigenvalue of the graph Laplacian

$$\gamma \lambda_2(\mathcal{G}) > 1 + \frac{1}{3}\beta_y^2 + \epsilon$$

- A Contraction Based Approach:
  - Yields a tighter bound

$$\gamma \lambda_2(\mathcal{G}) > 1$$



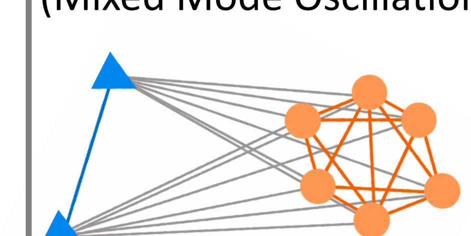


$a = 0.7, b = 0.8, \epsilon = 0.08, I = 0.5$

References:
[1] E. N. Davison, B. Dey, N. E. Leonard. *Synchronization Bound for Networks of Nonlinear Oscillators.* in Proceedings of Allerton Conference on Communication, Control, and Computing, 2016.

Future directions

(Mixed Mode Oscillation)



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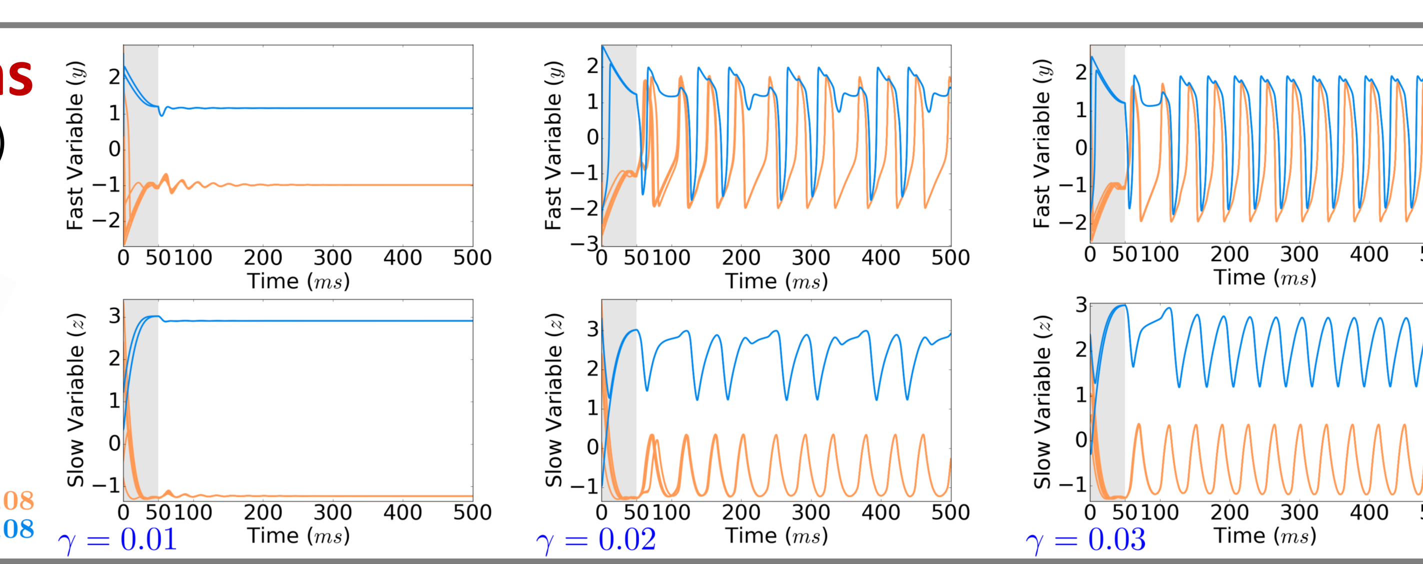

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