MAE 544 - Nonlinear Control

Fall 2017

Homework # 2

- Assigned: October 17, 2017.
- Due: Tuesday, October 24, 2017 by 3:00 pm in class.

1. *Let $\mathcal{M}_1$, $\mathcal{M}_2$ and $\mathcal{M}_3$ be smooth manifolds of dimension m_1, m_2 and m_3, respectively. Moreover, $\Phi : \mathcal{M}_1 \to \mathcal{M}_2$ and $\Psi : \mathcal{M}_2 \to \mathcal{M}_3$ are smooth maps. Then show that $\Psi \circ \Psi : \mathcal{M}_1 \to \mathcal{M}_3$ is a smooth map.*

2. *Consider the open ball $\mathcal{B}^n$ defined as $\mathcal{B}^n = \left\{(x_1, \dots, x_n) \in \mathbb{R}^n \middle| x_1^2 + \cdots + x_n^2 < 1\right\}$. Show that it is an n-dimensional smooth manifold. What is the least number of charts that one needs to construct an atlas for this manifold?*

3. *Consider the 2-Torus $\mathcal{T}^2 \subset \mathbb{R}^3$ defined as*

$$\mathcal{T}^2 = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 \middle| \left(\sqrt{x_1^2 + x_2^2} - R\right)^2 + x_3^2 = r^2\right\},$$

where $R > r > 0$. Show that $\mathcal{T}^2$ is a 2-dimensional smooth manifold.

4. *An n-dimensional smooth manifold $\mathcal{M}$ is said to be* parallelizable *if there exist smooth vector fields $X_1, \dots, X_n \in \mathfrak{X}(\mathcal{M})$ such that $X_1(p), \dots, X_n(p)$ define a basis of $T_p\mathcal{M}$ for all $p \in \mathcal{M}$. Let X_1, X_2 and X_3, defined as*

$$\begin{aligned}
X_1(x_1, x_2, x_3, x_4) &= x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2} + x_4 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_4} \\
X_2(x_1, x_2, x_3, x_4) &= x_3 \frac{\partial}{\partial x_1} - x_4 \frac{\partial}{\partial x_2} - x_1 \frac{\partial}{\partial x_3} + x_2 \frac{\partial}{\partial x_4} \\
X_3(x_1, x_2, x_3, x_4) &= x_4 \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_3} - x_1 \frac{\partial}{\partial x_4},
\end{aligned}$$

be three vector fields on the 3-sphere $\mathcal{S}^3 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \big| x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\}$. Show that $\mathcal{S}^3$ is parallelizable (i.e. $X_i, i \in \{1, 2, 3\}$ are smooth vector fields, and they define a basis for $T_p\mathcal{S}^3$ at every $p \in \mathcal{S}^3$).

5. *Consider the 2-Sphere $\mathcal{S}^2 \subset \mathbb{R}^3$ defined as*

$$\mathcal{S}^2 = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 \middle| x_1^2 + x_2^2 + x_3^2 = 1\right\}.$$

Construct an atlas for the associated tangent bundle $T\mathcal{S}^2$.