Lecture 1

MAE 544
Nonlinear Control

09/14/17.
BD
①

Control and Feedback

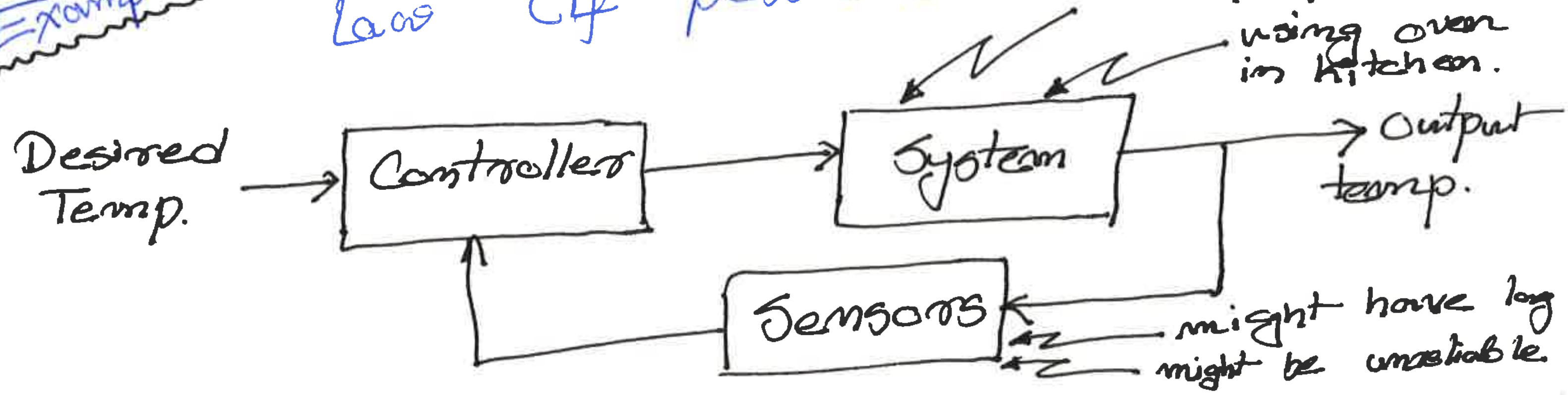
- Develop algorithms and feedback laws which aims to govern system behavior (e.g. physical/biological/financial)

- However, the ambient environment might not remain stationary; that's why it might be better to incorporate the current behavior while planning the next action/input. → FEEDBACK!!

- Feedback is everywhere (air conditioning, even when I am interacting with someone)

- In this course we will study how to analyze nonlinear systems and how to design appropriate control law (if possible).

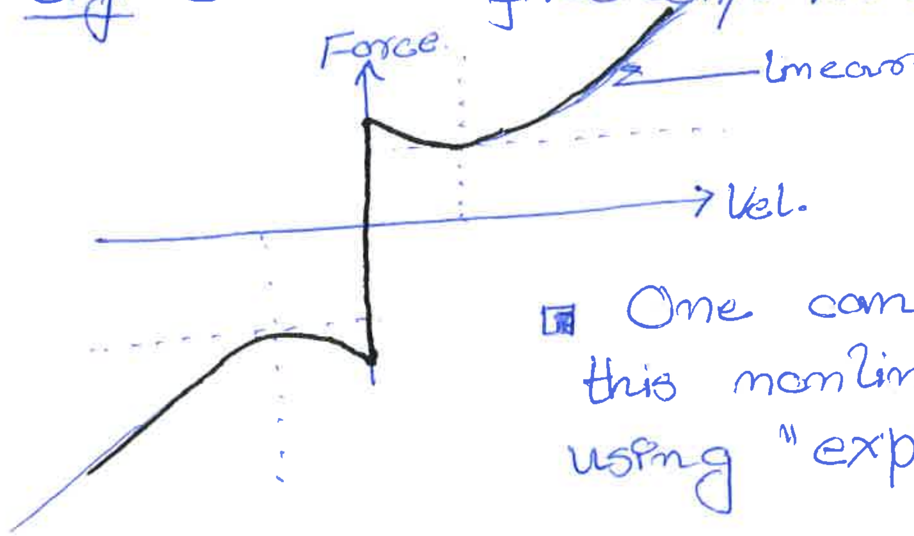
Example



Why nonlinear control?

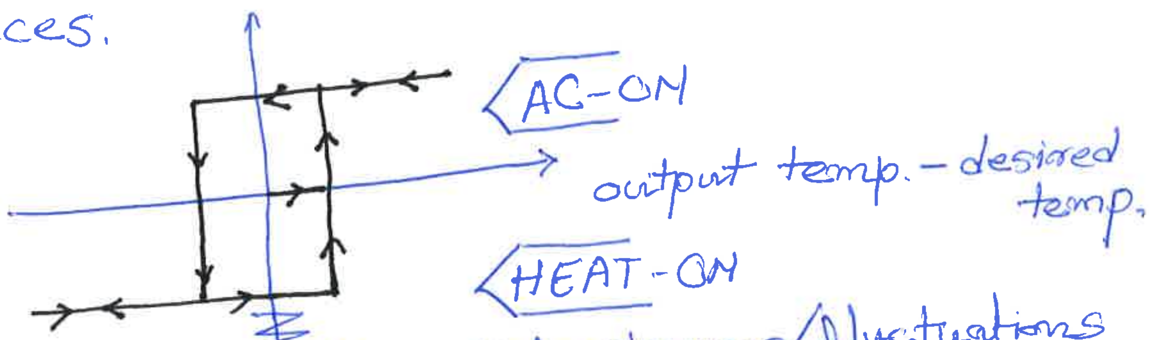
→ Real world systems are almost always nonlinear. (Superposition fails!) ← Freq. dom. techniques fail

e.g. Coulomb friction/Stribeck friction



One can model this nonlinear system using "exp", "tanh" etc.

→ Sometimes we use nonlinearities to improve robustness against small disturbances.



disturbances/fluctuations like this won't cause

otherwise we could have got chattering effect!

→ And sometimes, richness of behavior due to nonlinearities can be leveraged to design better controllers!

Consider the system:

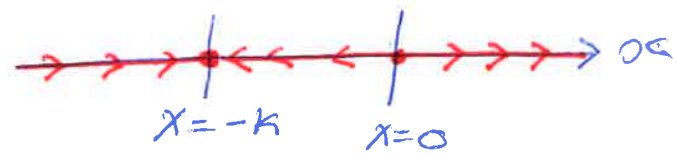
$$\boxed{\dot{x} = x^2 + u}$$

⊙ Can we design a control law, i.e.

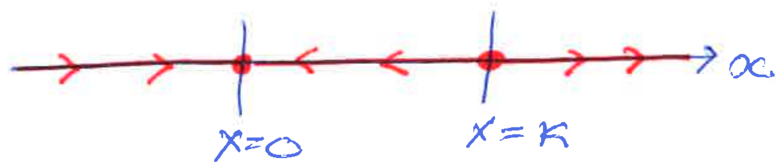
$$u = f(x),$$

s.t. trajectories of this system converges to the origin $(x=0)$ irrespective of their starting point?

Linear $\longrightarrow$ $u = kx \Rightarrow \dot{x} = x^2 + kx = x(x+k)$ $k>0$



$u = -kx \Rightarrow \dot{x} = x^2 - kx = x(x-k)$



n/l $\longrightarrow$ $\boxed{u = -x^2 - \alpha x}$, $\alpha > 0$

$\Rightarrow \dot{x} = x^2 - x^2 - \alpha x \Rightarrow \dot{x} = -\alpha x$

$\Downarrow$

$x(t) = x(0)e^{-\alpha t}$

(!!) A nonlinear control is the only option!

Examples from Space Missions:-

→ Newton's and Kepler's laws are the key contributors for orbital mechanics (Interesting to note that one can be derived from the other).

→ n-Body problem

Gravitational Constant

$$\boxed{\ddot{r}_i = G \sum_{j=1}^{m} m_j \frac{(r_j - r_i)}{\| r_j - r_i \|^3}} \quad i \neq j$$

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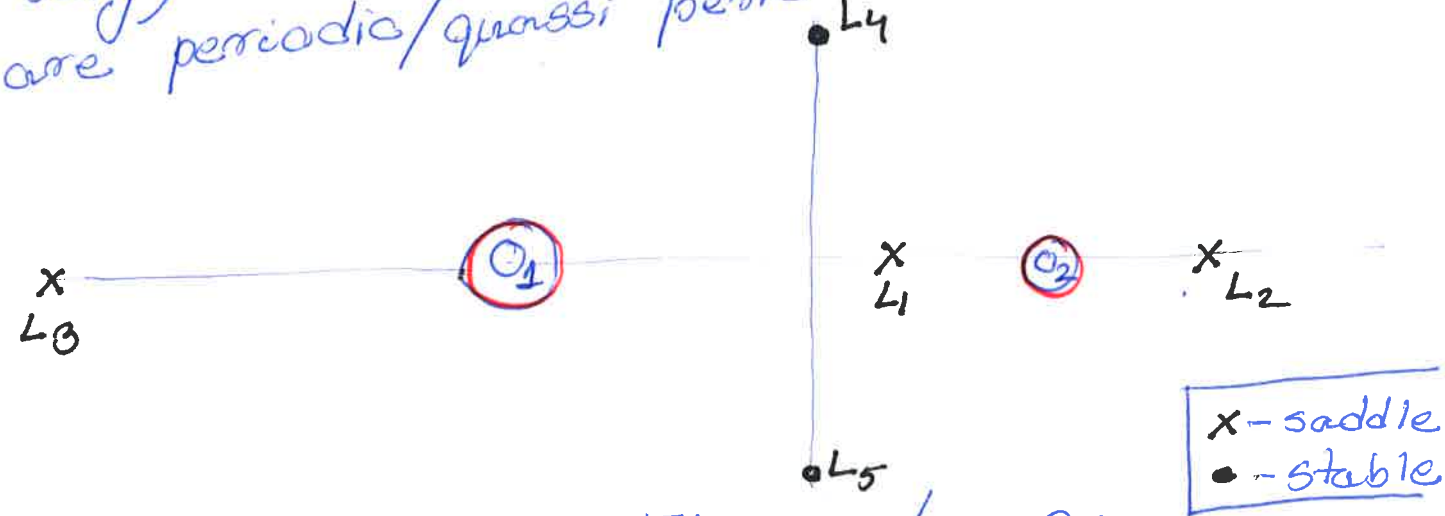
(III) For a 3-Body system we can get 5 locations where all the forces (gravitational pull from 2 large bodies and centrifugal force of the minor body (also coriolis)) balances each other. Also, there are periodic/quassi periodic orbits.



→ JamesWebb Space Telescope/WFIRST will orbit around L_2 (Halo orbit)

→ Solar observatory (SOHO) orbits around L_1.

→ Some of these orbits are stable which means less fuel and longer mission time!

(IV) Example from Epidemiology:-

→ SIR (Susceptible - Infected - Recovered) or SIS models and their variations.

SIS ↳ common cold

↓ flu/chicken pox

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→ Vaccinations

- Total population → N ← remains constant.
- Susceptible pop. → S
- Infected " → I
- Vaccinated " → V

$(S+I+V+R) = N$ ↑ fixed

- Constant birthrate → μ
- Rate of vaccination → $P \in [0,1]$

• Only newborns are vaccinated
• Vaccines give lifelong immunity

$$\dot{S} = \frac{dS}{dt} = \underbrace{\mu N(1-P)}_{\text{Birth}} - \mu S - \underbrace{\beta\left(\frac{I}{N}\right)S}_{\text{Infection}}$$

$$\dot{I} = \frac{dI}{dt} = \underbrace{\beta\left(\frac{I}{N}\right)S}_{\text{Infection}} - \mu I - \underbrace{\gamma I}_{\text{Recovery}}$$

$$\dot{V} = \frac{dV}{dt} = \underbrace{\mu N P}_{\text{Birth}} - \mu V$$

$$\dot{R} = \frac{dR}{dt} = \underbrace{\gamma I}_{\text{Recovery}} - \mu R$$

Death

Then one can ask how to choose $P \in [0,1]$ so as to make —

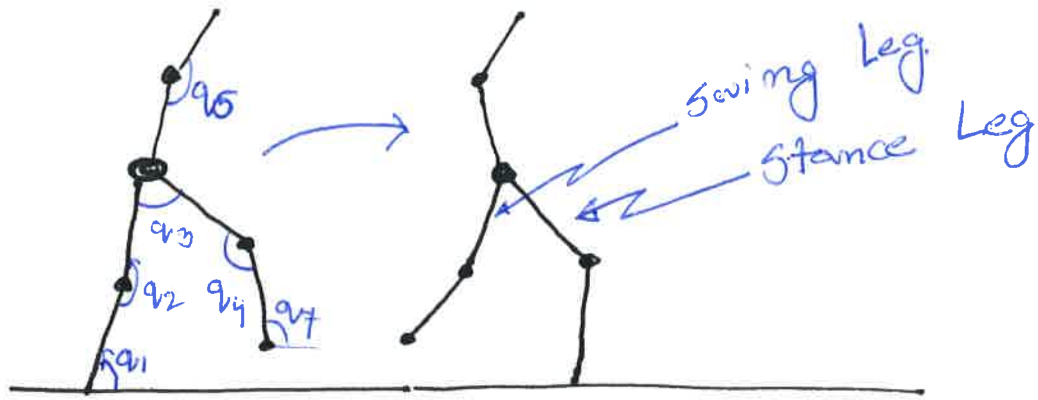
$$\boxed{\lim_{t\to\infty} I(t) = 0}$$ Disease eradication

Reqd. condition (Sufficient) $\boxed{P \geq 1 - \frac{\gamma}{\beta}}$ — $\gamma \uparrow \Rightarrow$ threshold ↓

$\beta \uparrow \Rightarrow$ threshold ↑

Example from Robotics —

→ Bipedal robots are critical for a variety of applications (e.g. search & rescue in a debris field).



→ Stacking all these joint angles "q_i" inside a single vector "q", its dynamics can be expressed as —

$$\boxed{D(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = B(q)u}$$

Other legged robots' dynamics can also be expressed in this way.

inertia matrix

damping

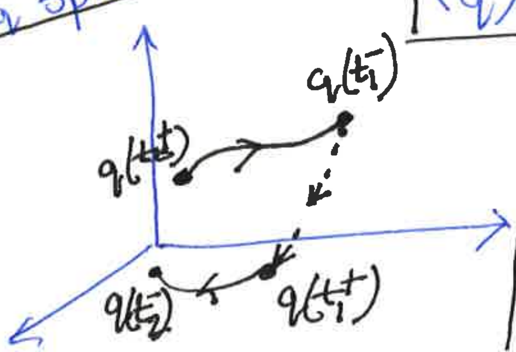
spring like force

torques applied at the joints.

→ When the swing leg hits the ground the roles are reversed, and velocities ($\dot{q}$) undergoes an instantaneous change.

$$\boxed{\begin{pmatrix} q \\ \dot{q} \end{pmatrix}^+ = \Delta \begin{pmatrix} q \\ \dot{q} \end{pmatrix}^-} \leftarrow \text{impact map.}$$

q-q̇ space.



→ continuous dynamics

--→-- impact map

To get stable gait (ie walking motion), one needs design $\boxed{u = f(q,\dot{q})}$ s.t. $q(t_{k+1}^-) = q(t_k^+)$ & $\dot{q}(t_{k+1}^-) = \dot{q}(t_k^+)$.

Example from Social Network —

→ We, as social beings, are always influenced by our peers. However, we might form our peer groups by connecting with like minded people or neighbors.

→ Suppose $x_i \in [0,1]$ represent the opinion of individual "i" on a certain topic in a group/population of size N.

→ Additionally, some individual may have a very strong preference (let's call these individuals: leaders − S).

Flexibility

$$\boxed{\begin{aligned} \dot{x}_i &= \sum_{j \in \mathcal{N}_i} \alpha_i (x_j - x_i) + \mathbb{1}_S(i)\, u_i \\ \mathcal{N}_i &= \left\{ j \in \{1, \dots, N\} \,\middle|\, \|x_j - x_i\| \leq \delta \right\} \end{aligned}}$$

personal preference (if any)

Indicator function — $\mathbb{1}_S(i) = \begin{cases} 1 & i \in S \\ 0 & i \notin S \end{cases}$

Alternatively,

$$\boxed{\dot{x}_i = \sum_{j=1}^{N} \alpha_i (x_j - x_i)\, \Phi_i(x_j) + \mathbb{1}_S(i)\, u_i}$$

where, $\Phi_i(x_j) = \mathbb{1}_{\{x \,|\, \|x - x_i\| \leq \delta\}}(x_j)$ is nonlinear.

→ How shall we choose $S \subseteq \{1, \dots, N\}$ and $u_i, i \in S$ if we want to drive the group's opinion to certain values or to polarize them?

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But cannot we achieve our goal by looking at the linearized dynamics ?

→ Results from linearization is not valid globally – they are local!

→ Sometimes results from linearization are not valid at all (e.g. $\dot{x}=x^2$, linearization around origin gives $\boxed{\delta\dot{x}=0}$ where δx is a small perturbation around origin)

→ Also, the underlying space might not equivalent to $\mathbb{R}^n$. (e.g. for an inverted pendulum, its state has a component which is an angle). So we might want to exploit coordinate - independent properties.

State-Space Models:—

State: $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$ ← An n-dimensional vector

↪ For Lagrangian systems, $x=(q,\dot{q})\in\mathbb{R}^{2N}$ if $q\in\mathbb{R}^N$

↪ Number of 1st order ODES.

↪ They provide a complete description about evolution of the system.

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State Equations:

$$\boxed{\dot{x} = f(t, x), \quad x(0) = x_0}$$

time-varying (nonautonomous)

$f: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$

$$\boxed{\dot{x} = f(x), \quad x(0) = x_0}$$

time-invariant (autonomous)

$f: \mathbb{R}^n \to \mathbb{R}^n$

→ If the evolution of the state is influenced by a control input —

$$\boxed{\dot{x} = f(x, u), \quad x(0) = x_0}$$

where $\underbrace{u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix} \in \mathbb{R}^m}_{\text{control i/p}}$

$f: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$

→ Now if we use a static feedback of the form: $u = \alpha(x)$, then the closed loop dynamics will look like —

$$\boxed{\dot{x} = f(x, \alpha(x)) = \tilde{f}(x)}$$

→ However, not every state may be accesible for designing the feedback. That leads to an output $y = \begin{bmatrix} y_1 \\ \vdots \\ y_p \end{bmatrix} \in \mathbb{R}^p$ defined as —

$$\boxed{y = h(x, u)}$$

$h: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$

For certain type of systems, we need $m = p$.

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→ Then, by using a static feedback of the form $u = \beta(y)$, the closed loop dynamics can be expressed as —

$$\boxed{\dot{x} = f\left(x, \beta\left(h\left(x, \beta(y)\right)\right)\right), x(0) = x_0}$$

$$= \hat{f}(x)$$

recursion

→ Thus, developing analysis tools for autonomous systems, helps us in studying closed-loop behavior of systems with input as well. But we also need constructive methods for finding an appropriate feedback law (α/β).

OUR FOCUS: $\boxed{\begin{aligned} \dot{x} &= f(x, u) \\ y &= h(x, u) \end{aligned}}$

- Now we go back to $\boxed{\dot{x} = f(x)}$ again.

→ $x^* \in \mathbb{R}^n$ is an equilibrium point iff $f(x^*) = 0$, i.e. equilibrium points are zero sets of $f: \mathbb{R}^n \to \mathbb{R}^n$.

→ $x^* \in \mathbb{R}^n$ is an isolated equilibrium point if there exists some $\delta > 0$ s.t. there is no other eq. inside the ball $B_\delta(x^*) = \{x \mid \|x - x^*\| \leq \delta\}$

→ **Linearization around x^*:**

Assuming $f: \mathbb{R}^n \to \mathbb{R}^n$ to be continuously differentiable, i.e. C^1, its linearization is given by —

$$\boxed{\dot{z} = \left(\frac{\partial f}{\partial x}\right)\Big|_{x^*} z}$$

Lemma: If $\left(\frac{\partial f}{\partial x}\right)_{x^*}$ is non-singular, then $x^* \in \mathbb{R}^n$ is an isolated equilibrium point.

↪ This is a sufficient condn, but not necessary! (e.g. $\dot{x} = x^3$)

Linearization around a trajectory for the system $\dot{x} = f(x,u)$: —

Consider $(x^*(t), u^*(t))$ be a solution trajectory for the system.

Then, $\dot{x}^*(t) = f(x^*(t), u^*(t))$

Assume, a perturbation of the input — from $u^*(t)$ to $u^*(t) + \delta u(t)$ leads to a perturbed trajectory $x^*(t) + \delta x(t)$.

Then,
$$\dot{x}^*(t) + \delta\dot{x}(t) = f(x^*(t) + \delta x(t), u^*(t) + \delta u(t))$$

Using Taylor Series expansion —

$$f(x^*(t)+\delta x(t), u^*(t)+\delta u(t))$$

$$= f(x^*(t), u^*(t)) + \left(\frac{\partial f}{\partial x}\right)\Big|_{(x^*(t),u^*(t))} \delta x(t) + \left(\frac{\partial f}{\partial u}\right)\Big|_{(x^*(t),u^*(t))} \delta u(t)$$

$$+ \text{higer order terms (h.o.t.)}$$

$$= \dot{x}^*(t) + \left(\frac{\partial f}{\partial x}\right)\Big|_{(x^*(t),u^*(t))} \delta x(t) + \left(\frac{\partial f}{\partial u}\right)\Big|_{(x^*(t),u^*(t))} \delta u(t)$$

$$+ \text{h.o.t.}$$

As $\delta\dot{x}(t) = f(x^*(t)+\delta x(t), u^*(t)+\delta u(t)) - \dot{x}^*(t)$, we can express the linearized dynamics around $(x^*(t), u^*(t))$ as —

$$\boxed{\delta\dot{x}(t) = A(t)\,\delta x(t) + B(t)\,\delta u(t)}$$

Linear Time-Varying System

where,

$$A(t) \triangleq \left(\frac{\partial f}{\partial x}\right)\Big|_{(x^*(t),u^*(t))}$$

&, $$B(t) = \left(\frac{\partial f}{\partial u}\right)\Big|_{(x^*(t),u^*(t))}$$