Lecture 2

MAE 544
NONLINEAR CONTROL

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Group: A group is a set (G) equipped with a binary operation ($+$) such that the following properties hold true —

i) For all $x, y \in G$, $x+y \in G$ (Closure)

ii) For all $x, y, z \in G$, $(x+y)+z = x+(y+z)$ (Associativity)

iii) There exists an element $e \in G$ such that $x+e = e+x = x$ for all $x \in G$. (Identity element)

iv) For all $x \in G$, there exists an element $\tilde{x} \in G$ such that $x+\tilde{x} = \tilde{x}+x = e$. This element $\tilde{x}$ is called the inverse of x. (Inverse) ☐

↳ Additionally, if $x+y = y+x$ for all $x, y \in G$, then G is called an abelian group.

↳ Claim: For any $x \in G$, its inverse is unique.

Proof: Suppose $x \in G$ has two inverses $\tilde{x}$ and $\hat{x}$.

Then, $\tilde{x} = \tilde{x} + e = \tilde{x} + (x + \hat{x}) = (\tilde{x} + x) + \hat{x}$
$= e + \hat{x} = \hat{x}$ ☐

Examples:

→ Integers with addition.

→ Matrices of size $m \times n$ with addition.

→ $n \times n$ non-singular matrices with multiplication

→ permutation group (has a connection with controllability for certain type of systems)

→ roots of unity with multiplication.

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Ring: A ring is a group with some additional structures imposed on it. A ring is a set (R) equipped with two binary operations ($+, \cdot$) such that the following properties hold true —

i) R is an abelian group under "+" with e as identity.

ii) For all $x, y, z \in R$, $x \cdot (y \cdot z) = (x \cdot y) \cdot z$. (Associativity)

iii) There exists an element $\mathbb{1} \in R$ such that $x \cdot \mathbb{1} = \mathbb{1} \cdot x = x$ for all $x \in R$. (Multiplicative identity)

iv) $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ and $(a + b) \cdot c = (a \cdot c) + (b \cdot a)$ for all $a, b, c \in R$. (Distributivity)

↳ Claim: For any R, its multiplicative identity is unique.

Proof: EXERCISE!

↳ Claim: For any $x \in R$, $x \cdot e = e = e \cdot x$ and $\hat{\mathbb{1}} \cdot x = \hat{x}$ where $\hat{\mathbb{1}} + \mathbb{1} = \mathbb{1} + \hat{\mathbb{1}} = e$ and $x + \hat{x} = \hat{x} + x = e$.

Proof: EXERCISE!

Examples:

→ Integers with addition and multiplications.

⊛→ Polynomials with addition and multiplications.

→ Power sets.

→ (has application in controllability results)

Field: A field is a ring with additional structure imposed on it. A set (F) equipped with two binary operations ($+, \cdot$) will be called a field if the following holds true —

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i) F is a ring with "+" and "$\cdot$", with e and $\mathbb{1}$.

ii) For any $x, y \in F$, the multiplication ($\cdot$) is commutative, i.e. $x \cdot y = y \cdot x$. (Commutativity)

iii) For every $x \in F$, $x \neq e$, there exists another element $y \in F$ s.t. $x \cdot y = y \cdot x = \mathbb{1}$. This element is called the multiplicative inverse of x. (Inverse) We typically use x^{-1} to denote multiplicative inverse of x.

↳ This is the structure that has more relevance in the initial part of this course.

↳ We also need a field in order to define a vector space.

Examples:

→ Rational numbers / Real numbers / Complex numbers.

→ $\mathbb{Z}_p$ i.e. the set of Integers modulo "p" when p is prime.

→ Rational fractions, i.e. ratio of polynomials.

→ ~~Smooth (infinitely differentiable) functions with positive range.~~ ~~Matrices~~

↳ Claim: If for $x, y \in F$, $x \cdot y = 0$ then either of x or y, or both must be zero.

Proof: Suppose $x \neq 0$.

Then, its multiplicative inverse x^{-1} exists.

$$0 = x^{-1} \cdot 0 = x^{-1} \cdot (x \cdot y) = (x^{-1} \cdot x) y = \mathbb{1} \cdot y = y. \quad \blacksquare$$

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Vector Space:

A vector space (V) over a field F is a set equipped two operations "$+$" (vector addition) and "$\cdot$" (scalar multiplication such that the following axioms hold true for any $x, y, z \in V$ and $\alpha, \beta \in F$ —

i) $x + y \in V$
ii) $\alpha x \in V$

} Closure

(As we will see later this plays a critical role in showing whether a space is subspace)

iii) $x + (y + z) = (x + y) + z$ (Associativity)

iv) $x + y = y + x$ (Commutativity)

v) There exists $0 \in V$ (called the zero vector) s.t. $x + 0 = 0 + x = x$

vi) For every x, there exists a unique element $(-x)$ such that $x + (-x) = (-x) + x = 0$. $(-x)$ is called the additive inverse of x.

vii) $\alpha(\beta x) = (\alpha\beta)x$

viii) $1x = x$ where 1 is the multiplicative identity of F.

ix) $(\alpha + \beta)x = \alpha x + \beta x$

x) $\alpha(x + y) = \alpha x + \alpha y$

↳ From axioms (iii) - (vi) we can conclude that a vector space is an abelian group.

↳ Vector subtraction and division by non-zero scalar can be defined.

Examples:

- n-tuple of real numbers, i.e. an ordered list of n-real numbers. We denote this space as $\mathbb{R}^n$.
- Similarly, $\mathbb{C}^n$ over $\mathbb{R}$ or $\mathbb{C}$, $\mathbb{Q}^n$ over $\mathbb{Q}$.
- $\mathbb{Z}_p^n$ over $\mathbb{Z}_p$ where p is a prime number.
- Real matrices of size $m \times n$, i.e. $\mathbb{R}^{m \times n}$, over $\mathbb{R}$.
- Continuous functions over $\mathbb{R}$.
- Solutions of $Ax = 0$ where $A \in \mathbb{R}^{m \times n}$ is given.

Linear independence and Basis:

- For a given vector space V, a set of vectors $v_1, v_2, \ldots, v_m \in V$ are <u>linearly independent</u> if $\sum_{i=1}^{m} \alpha_i v_i = 0$ ($\alpha_i \in F$, the underlying field) implies that $\alpha_i = 0$ for every $i \in \{1, \ldots, m\}$.
- Given a set of vectors $v_1, \ldots, v_m \in V$, its <u>span</u> is defined as the set of linear combinations, i.e.
$$\text{span}(\{v_1, \ldots, v_m\}) = \left\{ \sum_{i=1}^{m} \alpha_i v_i \;\middle|\; \alpha_i \in F, \ i = 1, \ldots, m \right\}$$
- A set of vectors $S = \{v_1, \ldots, v_m\}$ will be called a <u>basis</u> for the vector space V if $v_1, \ldots, v_m$ are <u>linearly independent</u> and <u>$\text{span}(S) = V$</u>.
- <u>Claim</u>: Any vector $v \in V$ can be uniquely represented as a linear combination of its basis vectors.

An example:

Consider the set of continuous functions defined over $[0, 2\pi]$.

$$V = \{ f : [0, 2\pi] \to \mathbb{R} \mid f : \text{continuous} \}$$

Clearly, $\cos x, \sin x \in V$ for $x \in [0, 2\pi]$.

*Are they linearly independent?

$$\alpha \cos x + \beta \sin x \equiv 0$$

$$\Rightarrow \sqrt{\alpha^2+\beta^2} \cos(x \mp \phi) \equiv 0 \quad \text{where } \phi = \operatorname{atan2}(\beta, \alpha)$$

$$\Rightarrow \sqrt{\alpha^2+\beta^2} = 0$$

$$\Rightarrow \alpha = \beta = 0$$

$\longrightarrow$ Hence, $\cos x$ and $\sin x$ are linearly independent.

→ A vector space V is finite dimensional if its basis set S has finite number of elements. Then, the cardinality of S will be called the dimension of V.

→ Given a vector space V over a field F, we can show that V is isomorphic to F^n where n is the dimension of V. (Hint: Think about the elements of V as linear combinations of its basis vectors) [Isomorphic means that there is an one-to-one and onto mapping between them.

→ Subspace: A subset $W \subseteq V$ will be called a subspace of V (a vector space over F) if W itself is a vector space over F.

↳ Let, $V = \{ M \in \mathbb{R}^{n \times n} \}$

Clearly V is a vector space over $\mathbb{R}$.

Define, $W = \{ A \in V \mid A^T = -A \}$ ← the set of $n \times n$ skew-symmetric matrices.

Then, W is a subspace of V.

We call this space "$so(n)$". It plays an important role in rigid body dynamics.

Inner products and Norms:

↳ Let, V be a vector space over F. An inner product is a map $\langle \cdot, \cdot \rangle : V \times V \to F$, $(v_1, v_2) \mapsto \langle v_1, v_2 \rangle$ such that —

i) $\langle v, w \rangle = \overline{\langle w, v \rangle}$ / $u, v, w \in V$; $\alpha, \beta \in F$

ii) $\langle \alpha v + \beta w, u \rangle = \alpha \langle v, u \rangle + \beta \langle w, v \rangle$

iii) $\langle v, v \rangle \geq 0$ and $\langle v, v \rangle$ if and only if $v = 0$.

The last property is called positive definiteness.

↳ Can be perceived as a generalization of dot products.

↳ Let, $V = \{ f : [0, 2\pi] \to \mathbb{R} \mid f : \text{continuous} \}$ ← i.e. the space of continuous functions over $[0, 2\pi]$

For $h, g \in V$ define —

$$\langle h, g \rangle = \int_0^{2\pi} h(x) g(x) \, dx$$

→ Clearly, $\langle h, g \rangle = \langle g, h \rangle$

→ $\langle \alpha h_1 + \beta h_2, g \rangle = \alpha \langle h_1, g \rangle + \beta \langle h_2, g \rangle$

→ $\langle h, h \rangle = \int_0^{2\pi} h^2(x) \, dx \geq 0$

and, $\langle h, h \rangle \stackrel{?}{=} \int_0^{2\pi} h^2(x) \, dx = 0 \Rightarrow h(x) \equiv 0$, i.e. $h = 0$

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↳ Let, V be a vector space equipped with $\langle \cdot, \cdot \rangle : V \times V \to F$. Then, a pair of vectors $v, w \in V$ are orthogonal (with respect to this inner product) if $\langle v, w \rangle = 0$.

↳ Claim: v, w are orthogonal $\Rightarrow$ they are linearly independent. The converse need not be true.

↳ Let, V be a vector space over F. A norm on V is a mapping $\|\cdot\| : V \to \mathbb{R}_+$ such that —

i) $\|x\| \geq 0$ for any $x \in V$ and $\|x\| = 0$ if and only if $x = 0$.

ii) $\|\alpha x\| = |\alpha| \|x\|$ for any $x \in V$ and $\alpha \in F$.

iii) $\|x + y\| \leq \|x\| + \|y\|$

↳ Given a vector space V with an inner product $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$, we can define a norm —

$$\|x\| = \sqrt{\langle x, x \rangle}$$ (induced from inner product)

↳ Cauchy - Schwarz Inequality:

$$|\langle x, y \rangle| \leq \|x\| \cdot \|y\| \quad \forall\ x, y \in V$$

→ It can be interpreted as a generalizn of $\boxed{x^T y = \|x\| \|y\| \cos \theta}$.

↳ Suppose, $v_1, v_2 \in V$ are linearly independent.

Define, $z = v_2 - \dfrac{\langle v_2, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1$

Then, $\langle v_1, z_1 \rangle = \overline{\left\langle v_1, v_2 - \frac{\langle v_2, v_1\rangle}{\langle v_1, v_1\rangle} v_1 \right\rangle}$

$$= \left\langle v_2 - \frac{\langle v_2, v_1\rangle}{\langle v_1, v_1\rangle} v_1, v_1 \right\rangle$$

$$= \overline{\langle v_2, v_1\rangle - \frac{\langle v_2, v_1\rangle}{\langle v_1, v_1\rangle}\langle v_1, v_1\rangle}$$

$$= 0$$

Also,

$$\alpha v_1 + \beta v_2 = \alpha v_1 + \beta\left[z_1 + \frac{\langle v_2, v_1\rangle}{\langle v_1, v_1\rangle} v_1\right]$$

$$= \left(\alpha + \frac{\beta\langle v_2, v_1\rangle}{\langle v_1, v_1\rangle}\right) v_1 + \beta z_1$$

Therefore, $\operatorname{span}(\{v_1, v_2\}) = \operatorname{span}(\{v_1, z_1\})$. Thus we can get a set of orthogonal vectors from a set of linearly independent vectors such that they have same span, i.e. they span the same subspace.

↳ $\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\right\} \longmapsto \left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$ ← An example

Sequence and Convergence:

↳ A sequence maps the set of natural numbers $\mathbb{N}$ to some suitable space X. (e.g.: $a_n = \frac{n+1}{n}$)

↳ Suppose X is a vector space equipped with a norm. Then a sequence $(a_n, n = 1, 2, 3, ---)$ converges to $a_0 \in X$ if for any $\epsilon > 0$, there exist $N(\epsilon)$ such that $\|a_n - a_0\| < \epsilon$ when $n \geq N(\epsilon)$.

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↳ A sequence $(a_n, n \in \mathbb{N})$, $a_i \in X$ is called a Cauchy sequence if for any $\varepsilon > 0$, there exists $N(\varepsilon)$ such that $\| a_n - a_m \| < \varepsilon$ whenever $m, n \geq N$.

→ Convergence implies Cauchy.

→ The converse is not true.

→ Counter example:

$X = \mathbb{Q}$ — the set of rational numbers.

$a_n = \frac{F_{n+1}}{F_n}$ — F_n is the n-th number in the Fibonacci seq.

As $F_n \in \mathbb{N}$, $a_n \in \mathbb{Q}$ for any $n \in \mathbb{N}$.

But $a_n \to a_0 = \frac{1+\sqrt{5}}{2} \notin \mathbb{Q}$

↳ A normed vector space X is complete if every Cauchy sequence in X converges to an element in X.

↳ A vector space V with an inner-product $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ will be called a Hilbert Space if this space is complete with respect to the norm induced by the inner-product.

→ Example: $\mathbb{R}^n$

↳ A vector space X with a norm will be called a Banach Space if this space is complete with respect to this norm.