Lecture 3

MAE 544

NONLINEAR CONTROL

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Lipschitz Continuity:

Lipschitz continuity is a stronger form of continuity. Or in other words, a Lipschitz continuous function is always continuous, but there are continuous functions which are not Lipschitz continuous.

→ A function $f: \mathbb{R}^n \to \mathbb{R}^m$ is Lipschitz continuous at $x \in \mathbb{R}^n$ if there exists a constant $M \geq 0$ and some suitable $\varepsilon > 0$ such that —

$$\| f(x_1) - f(x_2) \| \leq M \| x_1 - x_2 \|$$

for any $x_1, x_2 \in \mathcal{B}_\varepsilon(x)$ where $\mathcal{B}_\varepsilon(x) = \{ \tilde{x} \in \mathbb{R}^n \mid \| \tilde{x} - x \| < \varepsilon \}$ is an open ball around x.

→ Clearly, Lipschitz continuity is a local property (similar to continuity of a function)

→ A function $f: \mathbb{R}^n \to \mathbb{R}^m$ is Lipschitz continuous if there exists a constant $M \geq 0$ such that

$$\| f(x) - f(y) \| \leq M \| x - y \|$$

for any $x, y \in \mathbb{R}^n$.

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→ An interesting example:

$$\boxed{\begin{aligned} f : (0,\infty) &\to \mathbb{R} \\ x &\longmapsto \frac{1}{x} \end{aligned}}$$

Consider, the open ball $\mathcal{B}_\varepsilon(x)$ of radius $\varepsilon > 0$ and centered around $x \in (0,\infty)$. Then,

$$\mathcal{B}_\varepsilon(x) = \left\{ \tilde{x} \in (0,\infty) \,\middle|\, x - \varepsilon < \tilde{x} < x + \varepsilon \right\}$$

For any $x_1, x_2 \in \mathcal{B}_\varepsilon(x)$ —

$$\begin{aligned} \| f(x_1) - f(x_2) \| &= \left\| \frac{1}{x_1} - \frac{1}{x_2} \right\| \\ &= \left\| \frac{x_2 - x_1}{x_1 x_2} \right\| \\ &= \frac{\| x_1 - x_2 \|}{\| x_1 x_2 \|} \end{aligned}$$

As $x_1, x_2 > x - \varepsilon$, $\| x_1 x_2 \| > (x - \varepsilon)^2$

Therefore,

$$\| f(x_1) - f(x_2) \| \leq \left(\frac{1}{(x-\varepsilon)^2} \right) \| x_1 - x_2 \|$$

for any $x_1, x_2 \in \mathcal{B}_\varepsilon(x)$.

As a result $f : x \longmapsto \frac{1}{x}$ is Lipschitz continuous at any point in $(0,\infty)$. But it can be shown that this function is not Lipschitz continuous on $(0,\infty)$ as there does not exist any $M \geq 0$ such that —

$$\| f(x) - f(y) \| \leq M \| x - y \| \text{ for any } x, y \in (0,\infty).$$

→ Any function which is everywhere differentiable is Lipschitz continuous if and only if its first partial derivatives are bounded.

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→ Consider the function $f(x) = x^{1/3}$ defined on $\mathbb{R}$. This function is continuous; however, its first derivative $f'(x) = \frac{1}{3}x^{-2/3}$ is not bounded in any open neighborhood of $x = 0$. Hence, "f" is not a Lipschitz continuous function.

→ Lipschitz continuity plays a critical role in the existence of a unique solution for an ordinary differential equation.

→ If a function $f: \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz continuous with $0 \leq M < 1$ it is called a contraction mapping or contraction. In this case successive application of "f" will lead to eventual removal of any initial difference. Contraction plays a critical role in proving stability and synchronization in a nonlinear system. (Please have a look at the paper by Lohmiller and Slotine — uploaded on the course webpage).

Local Uniqueness:

Consider the system —

$$\dot{x}(t) = f(x(t)), \quad x(t) \in \mathbb{R}^n$$

and assume $f: \mathbb{R}^n \to \mathbb{R}^n$ to be Lipschitz continuous at $x_0 \in \mathbb{R}^n$, i.e. $\|f(x) - f(y)\| \leq M\|x-y\|$ for some $M \geq 0$ and for any $x, y \in \mathcal{B}_\varepsilon(x_0)$, $\varepsilon > 0$. Then, there exists some $\delta > 0$ such that, this system will have a unique trajectory/solution over the time interval $[t_0, t_0+\delta]$ starting at $x(t_0) = x_0$.

→ The proof of this theorem involves ideas about contraction mapping and completeness of normed spaces.

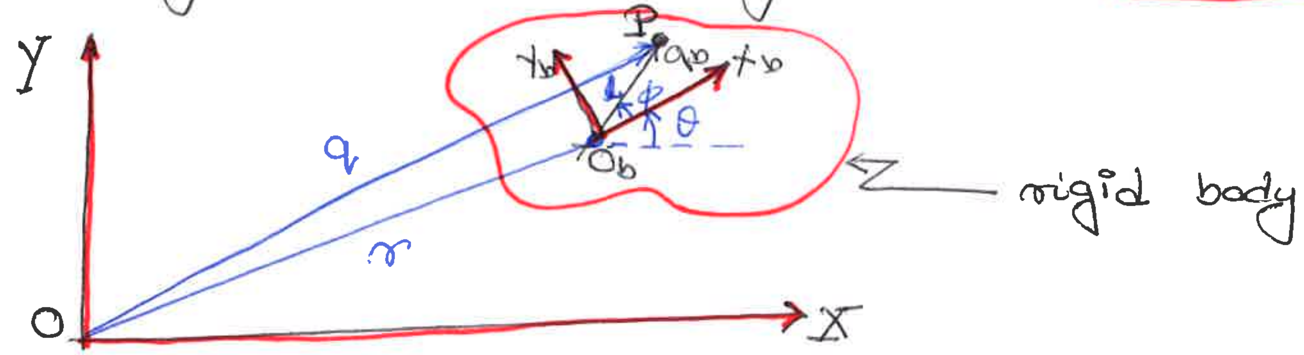
→ Consider the system —

$$\dot{x} = x^{1/3}, \quad x(0) = 0$$

→ Clearly $x(t) \equiv 0$, $t \geq 0$ is a solution trajectory of this system.

→ However, we can verify that $x(t) = \left(\frac{2t}{3}\right)^{3/2}$, $t \geq 0$ is also a solution of this system.

Rigid Body in a Planar setting



Let us consider a rigid body with a reference frame (X_b, Y_b) fixed to it.

$q_b = \begin{pmatrix} l \cos\phi \\ l \sin\phi \end{pmatrix} \leftarrow$ the position of point P with respect to the body fixed frame (X_b, Y_b)

$\boxed{\text{body-frame co-ords}}$

Let, (X, Y) define the laboratory/inertial reference frame.

Suppose the origin of the body frame (O_b) is located at position $r = \begin{pmatrix} r_x \\ r_y \end{pmatrix}$ in the inertial frame. Moreover, the body frame's orientation an angular offset of θ in the counter clockwise direction.

$q = \begin{pmatrix} q_x \\ q_y \end{pmatrix} \leftarrow$ position of point P with respect to the inertial frame

$\boxed{\text{inertial frame co-ords}}$

Then,
$$q = \begin{pmatrix} q_x \\ q_y \end{pmatrix} = \begin{pmatrix} r_x + l\cos(\theta+\phi) \\ r_y + l\sin(\theta+\phi) \end{pmatrix} = \begin{pmatrix} r_x \\ r_y \end{pmatrix} + \underbrace{\boxed{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}}}_{B(\theta)} \begin{pmatrix} l\cos\phi \\ l\sin\phi \end{pmatrix}$$

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Then,

$$\boxed{q = r + B(\theta)\, q_b}$$

← this gives us a rule to convert body frame co-ords to inertial frame co-ords.

→ Properties of $B(\theta)$

(i) → $B^T(\theta)B(\theta) = I_2$

(ii) → $\det(B(\theta)) = 1$

(iii) → $B(\theta_1+\theta_2) = B(\theta_1)B(\theta_2) = B(\theta_2+\theta_1) = B(\theta_2)B(\theta_1)$

(iv) → $\langle Bx, By\rangle = x^T B^T B y = x^T y = \langle x, y\rangle$

(v) → $\|Bx - By\|_2 = \|x-y\|_2$

- The set $\{B(\theta) \mid -\pi < \theta \leq \pi\}$ is a group under matrix multiplication. Moreover, we can interpret it as the group of rotations, $SO(2)$. It follows from (iii) that this group is abelian. The group operation, i.e. the matrix multiplication, can be perceived as application of successive rotations in a plane, and (iii) implies that the order of rotations do not matter on a plane.

→ In a similar way, the pair $(r, B(\theta))$ can be used to capture rigid body motions on a plane. The set $\{(r, B(\theta)) \mid r \in \mathbb{R}^2, B(\theta) \in SO(2)\}$ forms the group $SE(2)$.

- $q = r + B(\theta) q_b$ can also be expressed in the following form —

$$\boxed{\underbrace{\begin{pmatrix} q \\ 1 \end{pmatrix}}_{\in \mathbb{R}^{3\times 1}} = \begin{pmatrix} B(\theta) & r \\ 0 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} q_b \\ 1 \end{pmatrix}}_{\in \mathbb{R}^{3\times 1}}}$$

This allows us to perceive $SE(2)$ as a subset in the set of 3×3 square matrices

→ Suppose $(r_1, B_1) \in \mathbb{R}^2 \times SO(2)$ gives the rule to convert body frame coordinates (q_b) into an intermediate frame coordinate (q_i).

Then,
$$q_i = r_1 + B_1 q_b$$

Also, $(r_2, B_2) \in \mathbb{R}^2 \times SO(2)$ gives the rule to convert the intermediate frame co-ords (q_i) into the inertial frame coords (q).

Then,
$$q = r_2 + B_2 q_i = r_2 + B_2 (r_1 + B_1 q_b)$$
$$= (r_2 + B_2 r_1) + (B_2 B_1) q_b$$

Thus the composition of (r_1, B_1) and (r_2, B_2), i.e. the successive coordinate transformations, can be captured by a single co-ord. transformation given by —

$$(r_2 + B_2 r_1, B_2 B_1) \in \mathbb{R}^2 \times SO(2).$$

However, in this case the order of transformations matter.

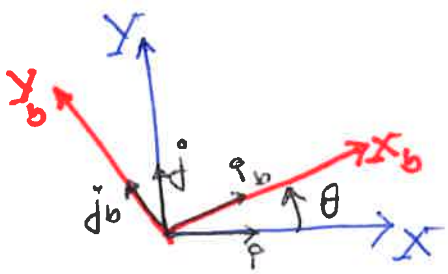
From the matrix perspective —

$$\begin{pmatrix} q \\ 1 \end{pmatrix} = \begin{pmatrix} B_2 & r_2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} q_i \\ 1 \end{pmatrix} = \begin{pmatrix} B_2 & r_2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} B_1 & r_1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} q_b \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} B_2 B_1 & B_2 r_1 + r_2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} q_b \\ 1 \end{pmatrix}$$

Extension to 3D setting

Suppose, (X_b, Y_b) is the body frame and (X, Y) is the intertial frame. Let, i_b, j_b be unit vectors along X_b, Y_b axes, and i, j be unit vectors along X, Y.



Then, $B(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$$= \begin{bmatrix} i \cdot i_b & i \cdot j_b \\ j \cdot i_b & j \cdot j_b \end{bmatrix}$$

This provides a means to express the rotation matrix "B" in terms of dot-products of unit vectors.

- In a 3D setting, let's assume that i_b, j_b, k_b and i, j, k represent the unit vectors along $X_b - Y_b - Z_b$ axes of the body frame and $X-Y-Z$ axes of the inertial frame. Furthermore, both frames are orthonormal and they share a common origin.

Then we can define B, which transforms body frame coordinates into inertial frame, as —

$$\boxed{B = \begin{bmatrix} i \cdot i_b & i \cdot j_b & i \cdot k_b \\ j \cdot i_b & j \cdot j_b & j \cdot k_b \\ k \cdot i_b & k \cdot j_b & k \cdot k_b \end{bmatrix}} \leftarrow \text{A } 3 \times 3 \text{ matrix}$$

→ We can show $\boxed{B^T B = I_3}$ and $\boxed{\det(B) = 1}$.

- Now assume that the origin of the body frame (X_b, Y_b, Z_b) is located at position $r \in \mathbb{R}^3$ in the inertial frame (X, Y, Z). Then inertial coordinate "q" of a point in the rigid can be expressed in terms of its body coordinate "q_b" as —

$$\boxed{q = r + B q_b}$$

→ Similar to the planar case successive coordinate transformations can be

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captured by a single transformation:

$$\boxed{(r_2, B_2), (r_1, B_1) \longmapsto (r_2 + B_2 r_1, B_2 B_1)}$$

(r_2, B_2): intermediate to inertial

(r_1, B_1): body to intermediate

$(r_2 + B_2 r_1, B_2 B_1)$: body to inertial.

→ Expressing (r, B) as $\begin{bmatrix} B & r \\ 0 & 1 \end{bmatrix}$ allows us to view it as a 4x4 matrix.

→ The set $\boxed{\{B \in \mathbb{R}^{3\times 3} \mid B^T B = I_3, \det(B) = 1\}}$ is a group under matrix multiplication. It is called the group of rotations in three dimensions and is expressed as SO(3). However, this is not an abelian group, i.e. order of rotations is important.

→ SO(3) is a subset within the 9-dimensional space of 3x3 matrices. But the property that $B^T B = I_3$ for any $B \in SO(3)$ gives rise to 6 algebraic constraints. As a result the dimension of SO(3) has dimension 3.

→ The set $\boxed{\{(r, B) \mid r \in \mathbb{R}^3, B \in SO(3)\}}$ is a group under multiplication. ~~However~~ This is not an abelian group, and we denote it as SE(3). Although, an element of this group can be expressed as a 4x4 matrix, dimension of SE(3) is 6.

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Parametrization of SO(3)

→ We were able to parametrize SO(2) using an angle θ, $\theta \in [-\pi, \pi]$, and SO(2) has a one-to-one and onto correspondence with the interval $[-\pi, \pi]$.

→ As SO(3) has dimension 3. Can we use a set of 3-parameters such that there is a one-to-one and onto mapping between those parameters and SO(3)?

- Answer is NO!!

Euler's Theorem: —

Any motion of a rigid body in a three dimensional setting with one point fixed (i.e. any element $B \in SO(3)$) can be obtained by a pure counter clockwise rotation by an angle 'ϕ' around axis "c" passing through the fixed point on the rigid body.

→ For a given $B \in SO(3)$ we can obtain ϕ and c as:

- $c \in \mathbb{R}^3$ satisfies $(I-B)c = 0$ ← eigen vector of B
- $\phi = \cos^{-1}\left(\frac{\text{trace}(B)-1}{2}\right)$

→ On the other hand, for a given $c \in \mathbb{R}^3$ and an angle ϕ, the corresponding rotation matrix $B \in SO(3)$ is given by —

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$$B = I + \hat{c}\,\sin\phi + \hat{c}^2(1-\cos\phi)$$

where,

$$\hat{c} = \begin{bmatrix} 0 & -c_3 & c_2 \\ c_3 & 0 & -c_1 \\ -c_2 & c_1 & 0 \end{bmatrix} \in \mathbb{R}^{3\times 3} \text{ for a given } c = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \in \mathbb{R}^3$$

skew symmetric matrix.

- Euler angles provide another alternative way to parametrize $SO(3)$.

However, none of these parametrizations are one-to-one.

Motion (Rotational Motion) of a Rigid Body:-

Let us consider a rigid body whose rotation and translation is represented by $(o, B) \in \mathbb{R}^3 \times SO(3)$ or in other words —

$$\boxed{q = B q_b}$$

gives us a rule to obtain inertial frame coordinates from the body frame coordinates.

Suppose, ω is the angular velocity of the rigid body in inertial frame coordinates.

Then, $\boxed{\dot{q} = \omega \times q = \hat{\omega} q}$ $\left(\hat{\omega} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \text{ for } \omega = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}\right.$

Moreover, as $B^T B = I$, the angular velocity for the body frame coordinates is : $\boxed{\Omega = B^T \omega}$

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On the other hand, as $q = Bq_b$ —

$$\dot{q} = \dot{B} q_b = \dot{B} B^T q$$

for a rigid body q_b is always fixed, i.e. $\dot{q}_b = 0$.

Therefore,

$$\dot{B} B^T = \hat{\omega} \implies \boxed{\dot{B} = \hat{\omega} B}$$

Also, $\dot{q} = \omega \times q$ can be expressed as —

$$\dot{B} q_b = B\Omega \times Bq_b = B(\Omega \times q_b) = B\hat{\Omega} q_b$$

Hence, $\boxed{\dot{B} = B\hat{\Omega}}$

Now, consider $J \in \mathbb{R}^{3\times 3}$ be the moment of inertia matrix in body coordinates.

Then,
angular momentum in body coordinates is —

$$\Pi = J\Omega,$$

and, in the inertial coordinate ~~it is~~ it can be expressed as — $\pi = B\Pi$

As the angular momentum does not change in absence of any external torque, its representation in the inertial frame will be fixed, i.e. —

$\dot{\pi} = 0$, leading to —

$$\dot{\pi} = B\dot{\Pi} + \dot{B}\Pi = B\dot{\Pi} + B\hat{\Omega}\Pi = B(\dot{\Pi} + \hat{\Omega}\Pi)$$
$$= B(\dot{\Pi} + \Omega \times \Pi)$$

Hence, $\boxed{\dot{\Pi} = -\Omega \times \Pi = \Pi \times \Omega}$