Lecture 4

MAE 544 NONLINEAR CONTROL

09/26/17

BD | 1

Earlier (in Lecture #2) we have seen that a finite dimensional vector space with a given basis set can be identified with the set of n-tuples ~~over F~~ over F, where "F" is the underlying field and "n" is the dimension of this vector space.

The idea of a manifold relates very well with this perspective towards vector space. Although the ~~ideas~~ concepts of vector addition and scalar multiplication are not carried over to this context.

Manifold:

A set $\mathcal{M}$ is called a manifold if each point $p \in \mathcal{M}$ has an open neighborhood which is homeomorphic to an open subset of $\mathbb{R}^n$ for some n.

→ Homeomorphism: A mapping $f: \mathcal{M}_1 \to \mathcal{M}_2$ will be called an homeomorphism if "f" is one-to-one and onto (i.e. a bijection), and both f and its inverse (f^{-1}) are continuous.

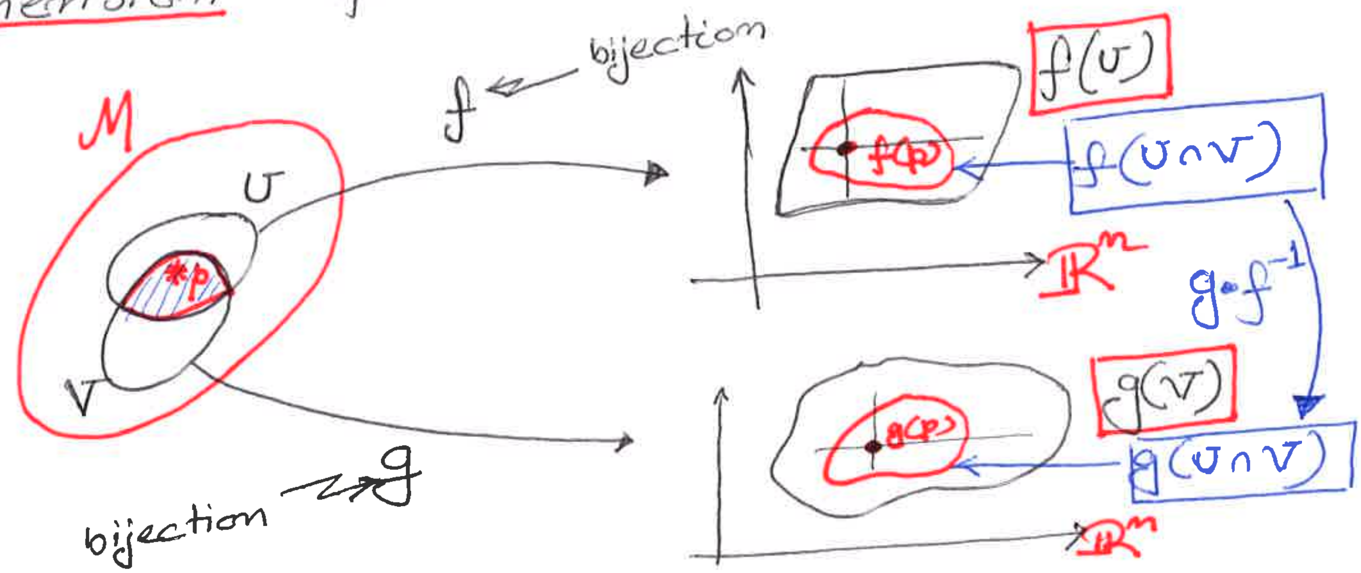
→ But what open neighborhood and continuity mean in this context? Do we need some notion of distance?

→ As it turns out, the idea of a topological space is very useful in this context.

09/26/2017
BD | 8-2

→ Lets assume that the manifold ($\mathcal{M}$) of our consideration lies within $\mathbb{R}^m$.

→ We used "n" in our definition of manifold to denote dimension of the underlying Euclidean set. If this number is fixed for any point on the manifold we call "n" to be the dimension of $\mathcal{M}$.



$$f: U \subseteq \mathcal{M} \longrightarrow f(U) \subseteq \mathbb{R}^n$$

$$p \longmapsto \begin{pmatrix} f^1(p) \\ | \\ f^n(p) \end{pmatrix} = \begin{pmatrix} x^1(p) \\ | \\ x^n(p) \end{pmatrix} \in \mathbb{R}^n$$

→ The neighborhood U of $p \in \mathcal{M}$ need not necessarily cover/include all of $\mathcal{M}$. So, there will be other such neighborhoods and associated functions. Each point of $\mathcal{M}$ must belong to atleast one such neighborhood.

→ (U, f) ~~is called~~ a called a chart. The collection of all such charts, such that any point of the manifold belongs to atleast one neighborhood, is called an atlas. 03/26/2017. 80 (4-3)

→ Any given point of the manifold $\mathcal{M}$, can belong to two different neighborhoods associated with two different charts. And, as the neighborhoods are open and their union covers the manifold, each neighborhood shall have non-zero intersection with some other neighborhood. Or in other words, each neighborhood must overlap with some other neighborhood.

Let, $V \subseteq \mathcal{M}$ be another neighborhood such that $p \in V$. Also (V, g) is a chart for the manifold $\mathcal{M}$.

So,
$$g : V \subseteq \mathcal{M} \longrightarrow g(V) \subseteq \mathbb{R}^n$$
$$p \longmapsto \begin{pmatrix} g^1(p) \\ | \\ g^n(p) \end{pmatrix} = \left.\begin{pmatrix} y^1(p) \\ | \\ y^n(p) \end{pmatrix}\right\} \text{Coordinates of } p \text{ under } g.$$

Consider the image of $U \cap V$ under f. We can define the following function Φ as

$$\Phi \triangleq g \circ f^{-1} : f(U \cap V) \subseteq \mathbb{R}^n \longrightarrow g(U \cap V) \subseteq \mathbb{R}^n$$
$$\begin{pmatrix} x^1 \\ | \\ x^n \end{pmatrix} \longmapsto \begin{pmatrix} y^1 \\ | \\ y^n \end{pmatrix}$$

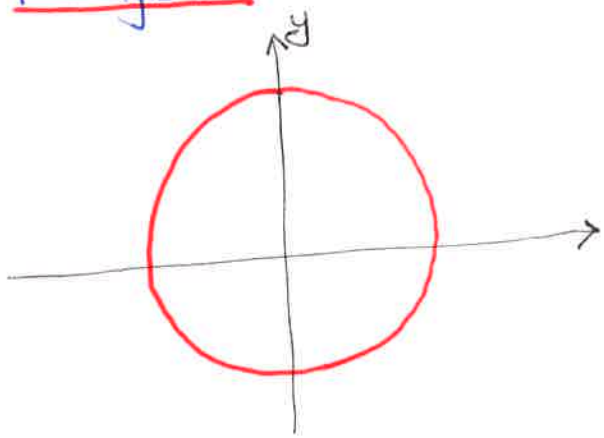
09/26/2017
BD1 (4-4)

→ Φ is a function from $\mathbb{R}^n$ to $\mathbb{R}^n$, and it is continuous. If partial derivatives of this function upto order k exists and are continuous, we call the charts (U, f) and (V, g) to be C^k-related. Moreover, when each point of M belongs atleast one neighborhood, and any two overlapping charts (i.e. the associated neighborhoods have non-trivial intersection) are C^k-related, we call the manifold M to be a C^k-manifold.

↪ $k = 1 \longrightarrow$ differentiable manifold.

↪ This is true for any k, $k = 0, 1, 2, \ldots$ ⟶ smooth manifold.

1-Sphere or a Circle as a 1-dimensional Manifold:—



$$S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$$

Define,

$$U = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1, y \neq 1\}$$

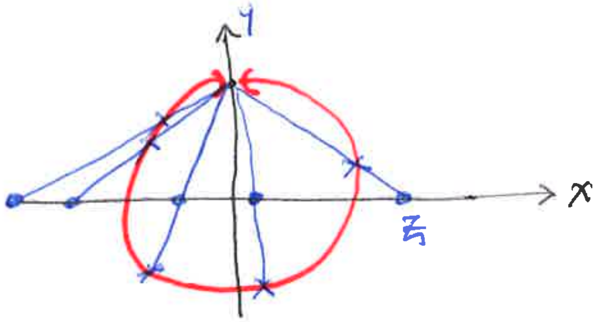
circle without north pole

$$V = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1, y \neq -1\}$$

circle without south pole.

$\boxed{\text{STEREOGRAPHIC PROJECTION}}$

09/26/2017
BD | 4-5

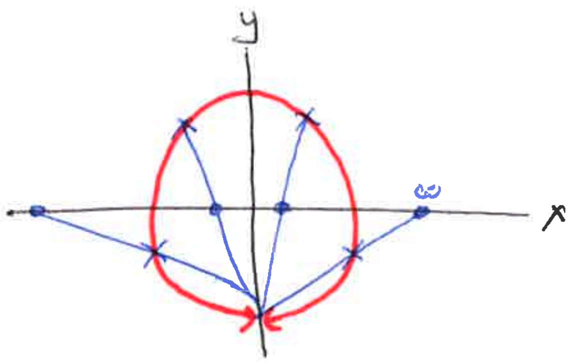


$f(x,y) = \dfrac{x}{1-y}$

- f is well defined on U
- f is continuous
- $f^{-1} : \mathbb{R} \to \mathbb{R}^2 \supseteq S^1$

  $z \longmapsto \left(\dfrac{2z}{z^2+1}, \dfrac{z^2-1}{z^2+1}\right)$

(U, f) is a chart.



$g(x,y) = \dfrac{x}{1+y}$

- g is well defined on V
- g is continuous
- $g^{-1} : \mathbb{R} \to \mathbb{R}^2 \supseteq S^1$

  $\omega \longmapsto \left(\dfrac{2\omega}{\omega^2+1}, \dfrac{1-\omega^2}{\omega^2+1}\right)$

(V, g) is a chart.

→ (U, f) and (V, g) form an atlas for the 1-sphere.

→ $g \circ f^{-1}$ is well-defined on the open interval $(-\infty, \infty) \setminus \{0\}$.

$$g \circ f^{-1} : (-\infty, 0) \cup (0, \infty) \longrightarrow \mathbb{R}$$
$$z \longmapsto \frac{1}{z}$$

→ We can similarly define $f \circ g^{-1}$.