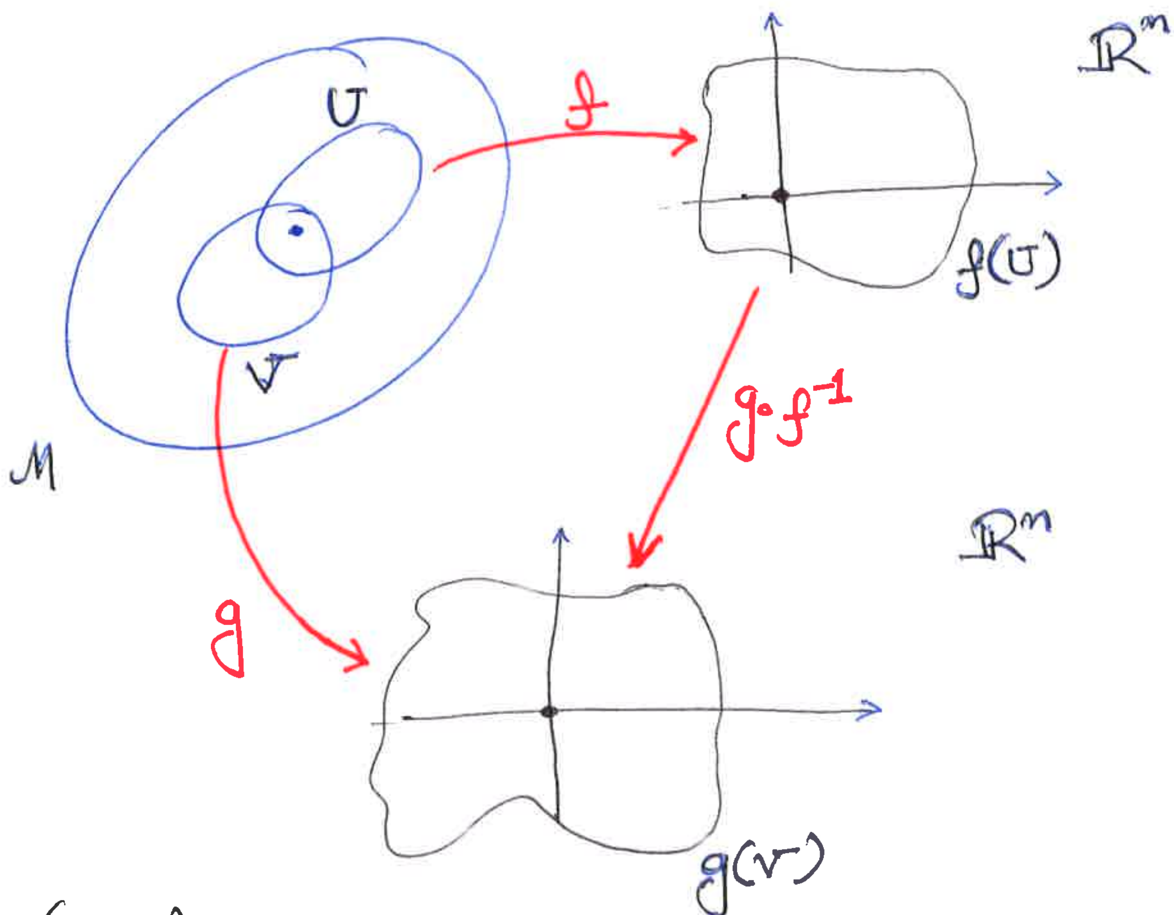
Lecture 5

MAE 544
NONLINEAR CONTROL

09/28/2017.
BD | 1

Recap on Manifold:—

→ A manifold locally behaves like an Euclidean space.



→ (U,f), (V,g) → charts ; f, g → lead to coordinate function.

→ If $p \in U$, then, $f(p) = \begin{bmatrix} f^1(p) \\ | \\ f^n(p) \end{bmatrix} \in \mathbb{R}^n$. These functions, i.e. $f_i : U \to \mathbb{R}^n$ are called coordinate functions. Thus a charts provides a local coordinate system.

→ An atlas $\mathcal{A}$ on M is a collection of charts such that the neighborhoods together cover

03/28/2017
BD | 5-3

the manifold $\mathcal{M}$.

$\boxed{\mathcal{A} = \{(U_\alpha, f_\alpha) \mid \alpha \in A\} \text{ such that } \mathcal{M} = \bigcup_\alpha U_\alpha}$

A: an appropriate index set (countable/uncountable)

→ Let us consider the function $g \circ f^{-1}$. This function is well-defined over $f(U \cap V) \subset \mathbb{R}^n$, and its range is given by $g(U \cap V)$. This function,

$$g \circ f^{-1} : \mathbb{R}^n \supset f(U \cap V) \to g(U \cap V)$$

is called a transition map.

→ In general, any two charts (U_α, f_α) and (U_β, f_β) where $\alpha, \beta \in A$ (the indexing set) are C^k-related if the transition maps

$$f_\alpha \circ f_\beta^{-1} : f_\beta(U_\alpha \cap U_\beta) \longrightarrow f_\alpha(U_\alpha \cap U_\beta)$$

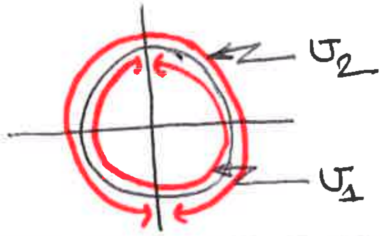
and, $f_\beta \circ f_\alpha^{-1} : f_\alpha(U_\alpha \cap U_\beta) \longrightarrow f_\beta(U_\alpha \cap U_\beta)$

have continuous partial derivatives upto order k.

→ If a manifold $\mathcal{M}$ can be equipped with an atlas $\mathcal{A} = \{(U_\alpha, f_\alpha) \mid \alpha \in A\}$, $\mathcal{M} = \bigcup_\alpha U_\alpha$, such that any pair of charts (U_α, f_α) and (U_β, f_β), $\alpha, \beta \in A$ are C^k-related, we call $\mathcal{M}$ to be a C^k-manifold and $\mathcal{A}$ to be its C^k-atlas.

→ Another important thing to note is that atlases can be combined together to give a new atlas.

→ Consider the example of S^1 :—



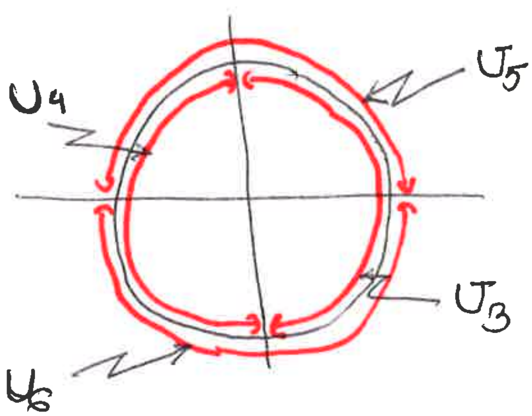
In last class we talked about two charts (U_1, f_1) and (U_2, f_2) where —

$$\boxed{S^1 = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2=1\}}$$

$$\boxed{\begin{aligned} U_1 &= S^1 \setminus \{(0,1)\} \\ U_2 &= S^1 \setminus \{(0,-1)\}, \end{aligned}}$$

and f_1 and f_2 were defined accordingly.

Clearly, $\mathcal{A} = \{(U_1, f_1), (U_2, f_2)\}$ is an atlas for S^1.



Now define,

$$\left[\begin{aligned} U_3 &= \{(x,y) \in S^1 \mid x > 0\} \\ f_3(x,y) &= y \end{aligned}\right.$$

$$\left[\begin{aligned} U_4 &= \{(x,y) \in S^1 \mid x < 0\} \\ f_4(x,y) &= y \end{aligned}\right.$$

$$\left[\begin{aligned} U_5 &= \{(x,y) \in S^1 \mid y > 0\} \\ f_5(x,y) &= x \end{aligned}\right. \qquad \left[\begin{aligned} U_6 &= \{(x,y) \in S^1 \mid y < 0\} \\ f_6(x,y) &= x \end{aligned}\right.$$

→ Clearly, $\hat{\mathcal{A}} = \bigcup_{k=\{3,4,5,6\}} (U_k, f_k)$ is also an atlas.

→ Moreover, $\hat{\hat{\mathcal{A}}} = \mathcal{A} \cup \hat{\mathcal{A}} = \bigcup_{k=1}^{6} (U_k, f_4)$ is also an atlas.

→ Now, if the union of two atlases is a C^k-atlas, then the original pair of atlases is called C^k-equivalent. Union of all atlases that are C^k-equivalent to each other, is called the maximal C^k-atlas. This is unique for a given manifold, and it defines a differential structure on M.

→ From here, we can guess that differentiability of a manifold does not depend on a particular choice for charts.

→ Also, maximal C^{k+1}-atlas $\subseteq$ maximal C^k-atlas and for any $k>0$, the maximal atlas of a C^k-manifold contains a C^∞-atlas. This follows from Whitney Extension Theorem. Due to this reason we will focus on C^∞ (smooth) structures on manifolds.

→ Smooth structures allow us to characterize functions defined on a smooth manifold.

→ Another important result from Whitney: Any m-dimensional manifold is contained (embedding sense) inside $\mathbb{R}^{2m}$ if the manifold is smooth.

→ Earlier we have shown how $SO(3)$ can be embedded in $\mathbb{R}^9$. But, this result allows us to embed it in $\mathbb{R}^6$.

08/28/2017
BD | 5-5

Smooth Structure on M:—

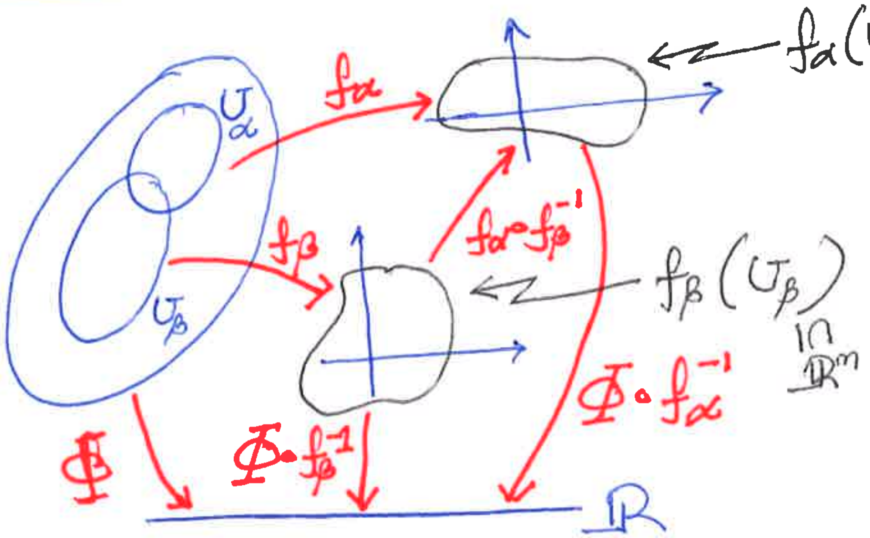
A smooth (C^∞) structure on a manifold M is a collection of charts $\{(U_\alpha, f_\alpha) \mid \alpha \in A\}$ such that —

i) $\bigcup_{\alpha\in A} U_\alpha = M$

ii) For any $\alpha, \beta \in A$, $f_\alpha \circ f_\beta^{-1} \in C^\infty$.

iii) The collection is maximal, i.e. any chart (V, g) such that $f_\alpha \circ g^{-1} \in C^\infty$, $g \circ f_\alpha^{-1} \in C^\infty$ $\forall \alpha \in A$ is contained in the collection.

Functions on M:—



→ M is a smooth manifold.

→ As a result it has an atlas $\mathcal{A} = \{(U_\alpha, f_\alpha) \mid \alpha \in A\}$ with $\bigcup_{\alpha\in A} U_\alpha = M$ such that $f_\alpha \circ f_\beta^{-1}$ is smooth for any $\alpha, \beta \in A$.

Consider,

$\Phi : M \to \mathbb{R}$ tobe a function defined on M.

→ Φ is smooth if $\Phi \circ f_\alpha^{-1} : \mathbb{R}^m \supseteq f_\alpha(U_\beta) \to \mathbb{R}$ is smooth for any $\alpha \in A$.

→ As $\Phi \circ f_\alpha^{-1}$ is a function from a subset of

09/28/2017
BD | 6-6

$\mathbb{R}^n$ to $\mathbb{R}$, its smoothness can be verified using calculus on $\mathbb{R}^n$.

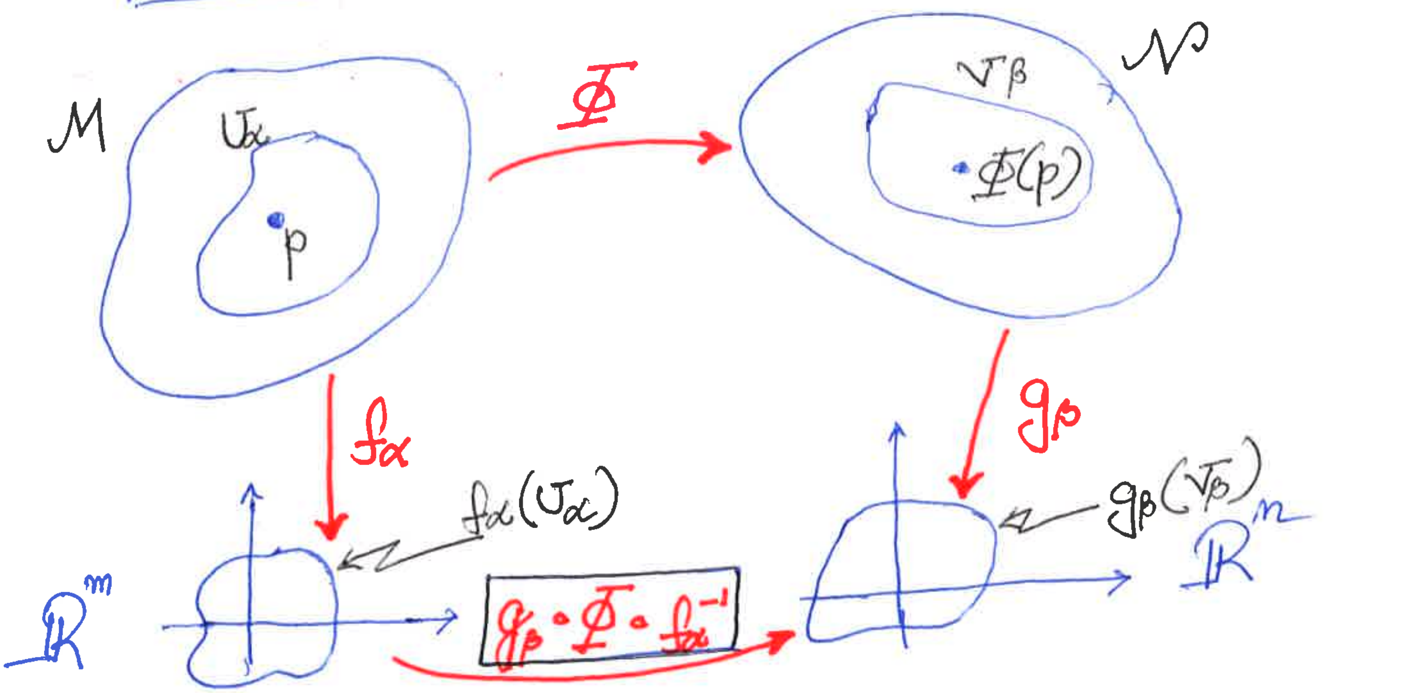
→ Now consider any two charts (U_α, f_α) and (U_β, f_β) such that $U_\alpha \cap U_\beta$ is non-empty. Then smoothness of $\Phi \circ f_\alpha^{-1}$ is equivalent to the smoothness of $\Phi \circ f_\beta^{-1}$. It directly follows from the fact that

$$\Phi \circ f_\beta^{-1} = \underbrace{\left(\Phi \circ f_\alpha^{-1}\right)}_{\text{smooth from assumption}} \circ \underbrace{\left(f_\alpha \circ f_\beta^{-1}\right)}_{\text{smooth from the def}^n\text{ of the smooth atlas}}$$

Smooth function between manifolds:—

→ Consider two smooth manifolds $\mathcal{M}$ and $\mathcal{N}$ and a function/mapping from $\mathcal{M}$ to $\mathcal{N}$.

$$\boxed{\Phi : \mathcal{M} \longrightarrow \mathcal{N}}$$



09/28/2017
BD (5-7)

→ As the m-dimensional manifold M is smooth it has a smooth structure associated with it —

— $\{(U_\alpha, f_\alpha) \mid \alpha \in A\}$ is a smooth atlas which is maximal.

→ In a similar way, the n-dimensional smooth manifold N has a smooth atlas $\{(V_\beta, g_\beta) \mid \beta \in B\}$ which is maximal.

→ These structures allow us talk about smoothness of $\Phi : M \to N$.

→ Φ is smooth at $p \in M$ if

$$g_\beta \circ \Phi \circ f_\alpha^{-1} : \mathbb{R}^m \supset f_\alpha(U_\alpha) \longrightarrow g_\beta(V_\beta) \subset \mathbb{R}^n$$
$$f_\alpha(p) \longmapsto g_\beta(\Phi(p))$$

is smooth. As $g_\beta \circ \Phi \circ f_\alpha^{-1}$ is a function over a subset of $\mathbb{R}^m$ its smoothness can be explored using calculus on $\mathbb{R}^m$.

→ Smoothness of Φ at $p \in M$ does not depend on a particular choice for the charts.

→ If Φ is smooth at any point $p \in M$, we call Φ to be a smooth function.

09/28/2017
BD | 5-8

Algebraic Manifold:—

→ Generalized the idea of smooth curves or surfaces to higher dimensions.

→ Consider the set —

$$\mathcal{M} = \left\{ (x_1, \text{—}, x_k) \in \mathbb{R}^k \,\middle|\, \begin{array}{l} f_i(x_1, \text{—}, x_k) = 0 \\ i = 1, 2, \text{--}, l \end{array} \right\}$$

Such that —

→ each function $f_i : \mathbb{R}^k \to \mathbb{R}^k$ is a polynomial.

→ f_i's are linearly independent.

Then we call $\mathcal{M}$ an algebraic variety. If, in addition, $\mathcal{M}$ is also an manifold, i.e. locally ~~homeomorp~~ homeomorphic to $\mathbb{R}^{k-l}$, $\mathcal{M}$ is called an algebraic manifold.

→ Example —

→ $S^2 = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 \,\middle|\, x_1^2 + x_2^2 + x_3^2 = 1 \right\}$

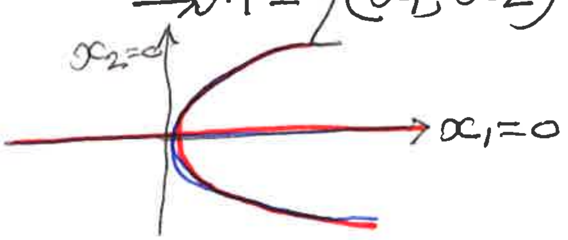
2-sphere

→ $\mathcal{M} = \left\{ (\underbrace{x_1, \text{—}, x_n}_{= x}) \in \mathbb{R}^n \,\middle|\, Ax = 0 \right\}$

- $A \in \mathbb{R}^{m \times n}$
- A has full row rank.

→ Consider —

→ $\mathcal{M} = \left\{ (x_1, x_2) \in \mathbb{R}^2 \,\middle|\, x_1^3 - x_1 x_2 = 0 \right\}$



- Tangent is not well defined at (0,0).
  — it is called singular pt.

09/28/2017
BD (5-9)

If an algebraic variety does not have any singular point then it is also a manifold.

Tangent Vectors on $\mathbb{R}^n$:—

— Let us focus on $\mathbb{R}^2$, and consider a particle moving on the plane.

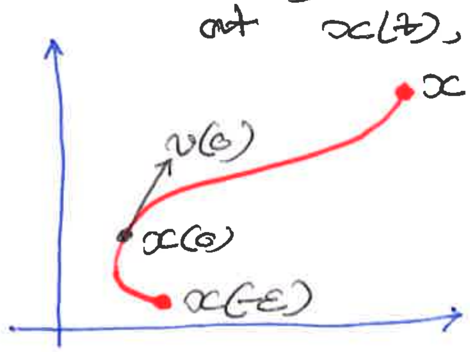
— Assume that its position at time t is given as $x(t)$ within the interval $(-\varepsilon, \varepsilon)$.

$$x : (-\varepsilon, \varepsilon) \longrightarrow \mathbb{R}^2$$

If $v(t)$ is its velocity at time "t" within the interval $(-\varepsilon, \varepsilon)$, then

$$v(t) = \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} = \frac{d}{dt}\left(\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}\right) = \frac{d}{dt} x(t) \qquad t \in (-\varepsilon, \varepsilon)$$

it is a vector with origin at $x(t)$, and tangent to ~~x(t)~~ the curve at $x(t)$.



• We call $v(0) = v(t)\Big|_{t=0}$ a tangent vector at $x(0)$.

tangent to the plane.

— Now consider a smooth function $f : \mathbb{R}^2 \to \mathbb{R}$

03/28/2017
BD1 (5-10)

Then, the derivative of $f:\mathbb{R}^2 \to \mathbb{R}$ along the curve $x(t), t \in (-\varepsilon, \varepsilon)$ at $t=0$ can be computed as —

$$\frac{d}{dt}\left(f\circ x(t)\right)\Big|_{t=0} \qquad \boxed{f\circ x : (-\varepsilon,\varepsilon)\to\mathbb{R}}$$

$$= \frac{\partial f}{\partial x_1}\Big|_{x=x(0)} \cdot \frac{\partial x_1}{\partial t}\Big|_{t=0} + \frac{\partial f}{\partial x_2}\Big|_{x=x(0)} \cdot \frac{\partial x_2}{\partial t}\Big|_{t=0}$$

$$= \frac{\partial f}{\partial x_1}\Big|_{x=x(0)} \cdot v_1(0) + \frac{\partial f}{\partial x_2}\Big|_{x=0} \cdot v_2(0)$$

$$= \left(v_1(0)\frac{\partial}{\partial x_1}\Big|_{x=x(0)} + v_2(0)\frac{\partial}{\partial x_2}\Big|_{x=x(0)}\right)(f)$$

Can be interpreted as a function on the space of smooth functions: $C^\infty \to \mathbb{R}$

$$:= v(0)(f)$$

directional derivative.

By interpreting $v(0)$ as a function over the space of smooth functions, i.e.

$$v(0): C^\infty \to \mathbb{R}$$
$$f \longmapsto v(0)(f)$$

we can notice that $v(0)$ is linear, i.e.

$$v(0)(c_1 f + c_2 g) = c_1\, v(0)(f) + c_2\, v(0)(g).$$

09/28/2017
BD| 5-11

Also, $v(0)$ satisfies Leibnetz rule, i.e.

$$v(0)(fg) = v_1(0)\left.\frac{\partial(f(x)g(x))}{\partial x_1}\right|_{x=x(0)} + v_2(0)\left.\frac{\partial(f(x)g(x))}{\partial x_2}\right|_{x=x(0)}$$

$$= v(0)(g)\cdot f(x(0)) + v(0)(f)\cdot g(x(0))$$

→ This provides us a way to think about tangent vectors as $\boxed{\text{derivations}}$ in the space of smooth functions.

It is a linear map over algebra (i.e. vector space with vector multiplication) which satisfies Leibnetz's law.