Lecture 6

# MAE 544
# NONLINEAR CONTROL

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BD | 1

## Derivation:

A vector space equipped with a bilinear product is called an algebra.

example ($\mathbb{R}^3$ with cross-product)

→ A derivation ($D$) over an algebra ($A$) is a mapping $D: A \to A$ such that —

i) $D(\alpha a + \beta b) = \alpha D(a) + \beta D(b)$ for any $a, b \in A$ and $\alpha, \beta \in K$, where $K$ is the underlying field.

ii) $D(ab) = D(a)b + D(b)a$ (Leibnetz rule)

In our case we have $v(o): C^\infty \to \mathbb{R}$ defined as

$$v(o)(f) = \frac{d}{dt}\left(f \circ x(t)\right)\Big|_{t=0}$$

where,

$x: (-\varepsilon, \varepsilon) \to \mathbb{R}^2$, and $v(o)$ is tangent to $x(t)$ at $t=0$, i.e. $v(o) = \begin{bmatrix} \frac{\partial x_1}{\partial t} & \frac{\partial x_2}{\partial t} \end{bmatrix}\Big|_{t=0}$.

10/05/2017
BD | 6+2

# Tangent vectors on Smooth Manifold:

→ A manifold $\mathcal{M}$ with a smooth structure imposed by $\{(U_\alpha, f_\alpha) / \alpha \in A\}$.

→ Consider a smooth curve $\boxed{\sigma : (-\varepsilon, \varepsilon) \to \mathcal{M}}$ passing through $p \in \mathcal{M}$, i.e $\sigma(0) = p$.

- Smoothness of the curve is equivalent to smoothness of the function —

$$f_\alpha \circ \sigma : \mathbb{R} \supset (-\varepsilon, \varepsilon) \longrightarrow f_\alpha(U_\alpha) \subseteq \mathbb{R}^n$$

where $U_\alpha$ is a neighborhood of $p \in \mathcal{M}$.

- Also, if $f_\alpha \circ \sigma$ is smooth for some $\alpha \in A$ such that $p \in U_\alpha$, then $f_\alpha \circ \sigma$ is smooth for all $\alpha \in A$ such that $p \in U_\alpha$.

→ Let, $\Phi : \mathcal{M} \to \mathbb{R}$. Then, $\Phi \circ \sigma$ is a real-valued function defined on $(-\varepsilon, \varepsilon)$, and we can define a tangent vector $v_p$ to $\mathcal{M}$ at a point $p \in \mathcal{M}$ as —

$$\boxed{v_p(\Phi) = \frac{d}{dt}\left(\Phi \circ \sigma(t)\right)\Big|_{t=0}}$$

- Suppose $\tilde{\sigma} : (-\varepsilon, \varepsilon) \to \mathcal{M}$ is another smooth curve on $\mathcal{M}$ such that $\tilde{\sigma}(0) = p \in \mathcal{M}$. It defines the same tangent vector $v_p$ if —

10/05/2017
BD | 6-3

$$\boxed{\left.\frac{d}{dt}\left(\Phi\circ\tilde{\gamma}\right)\right|_{t=0} = \left.\frac{d}{dt}\left(\Phi\circ\gamma\right)\right|_{t=0}}$$

for any smooth function $\Phi : M \to \mathbb{R}$. This implies that $\tilde{\gamma}$ and $\gamma$ are infinitesimally equivalent. Also, this defines an equivalence relationship, and as a result an tangent vector can be perceived as an equivalence class.

→ Similar to the planar case on $\mathbb{R}^2$, tangent vectors of a smooth manifold ($M$) can also be perceived as a derivation on the space of smooth functions on $M$.

⊙ Let, $M$ be a smooth manifold of dimension $n$, and $p \in M$. Then, a tangent vector at point $p \in M$ can be viewed as a derivation at point $p$, i.e. a smooth map $v_p : C^\infty(M) \to \mathbb{R}$ such that

— i) $v_p(\alpha\Phi + \beta\Psi) = \alpha v_p(\Phi) + \beta v_p(\Psi)$ for any $\Phi, \Psi \in C^\infty(M)$ and $\alpha, \beta \in \mathbb{R}$.

ii) $v_p(\Phi\Psi) = \Phi(p)\cdot v_p(\Psi) + \Psi(p)\, v_p(\Phi)$