Tangent Space on a Smooth Manifold:—

→ Let, M be a smooth manifold of dimension n, and $p \in M$ is a point on the manifold.

→ Then, a tangent vector v_p at point $p \in M$ can be viewed as a derivation at point p, i.e.

$v_p : C^\infty(M) \to \mathbb{R}$ is a smooth map such that —

$$v_p(\alpha\Phi + \beta\Psi) = \alpha v_p(\Phi) + \beta v_p(\Psi), \text{ and}$$

$$v_p(\Phi\Psi) = \Phi(p)\, v_p(\Psi) + \Psi(p)\, v_p(\Phi)$$

for any functions $\Phi, \Psi \in C^\infty(M)$ and $\alpha, \beta \in \mathbb{R}$.

→ The set of derivations at point p is called the tangent space at point p, and we denote it as T_pM. T_pM is a vector space of dimension n.

Coordinate Representation of Tangent Space:—

Let, (U, f) be a chart covering $p \in M$, i.e.

U is an open neighborhood of the point p.

Moreover, without loss of generality we can assume that, $f(p) = 0 \in \mathbb{R}^n$.

Now we consider a real-valued smooth function $\Phi : M \to \mathbb{R}$. Then, $\Phi \circ f^{-1} : \mathbb{R}^n \to \mathbb{R}$

Then, by defining

$$\chi_i : \mathbb{R}^n \to \mathbb{R}$$
$$(x_1, \ldots, x_n) \mapsto x_i,$$

we can introduce the coordinate function as —

$$x_i : M \to \mathbb{R}$$
$$p \mapsto \chi_i(f(p)).$$

Next, we define —

$$\frac{\partial}{\partial x_i}\bigg|_p (\Phi) \triangleq \frac{\partial}{\partial \chi_i}\left(\Phi \circ f^{-1}\right)(0) = \frac{\partial}{\partial \chi_i}\bigg|_0 (\Phi \circ f^{-1})$$

- $(\chi_1, \chi_2, \ldots, \chi_n)$ defines the coordinate system for $\mathbb{R}^n$ (i.e., the codomain of the coordinate chart map "f").

On the contrary $(x_1, \ldots, x_2)$ is the coordinate map for U under the action of f, i.e. $(x_1(p), \ldots, x_n(p))$ provides the local coordinate of $f(p)$ in $\mathbb{R}^n$.

It is easy to check that $\frac{\partial}{\partial x_i}\big|_p$ is an element of T_pM, the tangent space at $p \in M$.

Also, by setting $\Phi = x_i : M \to \mathbb{R}$, we have —

$$\frac{\partial}{\partial x_i}\bigg|_p (\Phi) = \frac{\partial}{\partial \chi_i}\left(\chi_i \circ f \circ f^{-1}\right)(0) = \frac{\partial}{\partial \chi_i}(\chi_i)(0) = 1$$

So, $\frac{\partial}{\partial x_i}\big|_p$ is nontrivial.

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◉ Previously, we have interpreted the elements of T_pM as tangents to appropriate smooth curves passing through $p \in M$.

For a smooth curve

$$\gamma : (-\varepsilon, \varepsilon) \to M, \quad \gamma(0) = p,$$

we defined the tangent vector $\gamma'(0)$ as—

$$\gamma'(0)(\Phi) = \frac{d}{dt}(\Phi \circ \gamma(t))\Big|_{t=0}$$

$$= \frac{d}{dt}\Big|_{t=0} (\Phi \circ \gamma)$$

where $\Phi \in C^\infty(M)$.

Now, define $\gamma : (-\varepsilon, \varepsilon) \to M$ such that

$$\gamma(t) = f^{-1}(0, \text{—}, t, \text{—} 0)$$

i-th position.

Then it readily follows that—

$$\boxed{\frac{\partial}{\partial x_i}\Big|_p = \gamma'(0),}$$

This provides a connection between the two alternative interpretations for tangent vectors.

◉ We can also show that the set $\left\{\frac{\partial}{\partial x_i}\Big|_p\right\}_{i=1}^{n}$ is linearly independent.

Proof: $\sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i}\Big|_p = 0 \Rightarrow \left(\sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i}\right)(\Phi) = 0 \quad \forall \Phi \in C^\infty(M)$

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Then, by setting $\Phi = x_i = \chi_i \circ f$ (i.e. the coordinate functions), we get

$$a_i = 0 \quad \text{for all } i = 1, \ldots, m.$$

This in turn proves that $\left\{ \frac{\partial}{\partial x_i}\Big|_p \right\}_{i=1}^m$ are linearly independent tangent vectors. ☒

→ For any tangent vector $v_p \in T_pM$, we can express it as —

$$v_p = \sum_{i=1}^m a_i \frac{\partial}{\partial x_i}\Big|_p,$$

and its coordinate representation can be written as $(a_1, \ldots, a_m)$. Moreover, these coordinate coefficients are given by —

$$a_i = v_p(x_i) = v_p(\chi_i \circ f).$$

→ Suppose that the point $p \in M$ belongs to another chart $(\tilde{U}, \tilde{f})$ as well. In this chart, $(y_1, \ldots, y_m)$ is the coordinate map for $\tilde{U}$ under $\tilde{f}$, and $(y_1(p), \ldots, y_m(p))$ provides the local coordinates of $\tilde{f}(p)$ in $\mathbb{R}^m$.

Then, $\left\{ \frac{\partial}{\partial y_i}\Big|_p \right\}_{i=1}^m$ provides a second set of basis vectors for T_pM. In what follows we explore how the coordinate representations of a given vector in these two basis sets (i.e., $\left\{ \frac{\partial}{\partial x_i}\Big|_p \right\}$ and $\left\{ \frac{\partial}{\partial y_i}\Big|_p \right\}$) are related to each other.

Since, $\frac{\partial}{\partial y_i}\Big|_p \in T_pM$, it can expressed in terms of the basis vectors $\left\{\frac{\partial}{\partial x_i}\Big|_p\right\}_{i=1}^{n}$ as —

$$\frac{\partial}{\partial y_i}\Big|_p = \sum_{j=1}^{n}\left(\frac{\partial}{\partial y_i}\Big|_p (x_j)\right)\frac{\partial}{\partial x_j}\Big|_p \quad \text{———} \quad (*)$$

Now, by letting $v \in T_pM$ be any given vector in the tangent space at $p \in M$, we can express v as —

$$v = \sum_{i=1}^{n}\alpha_i \frac{\partial}{\partial x_i}\Big|_p = \sum_{j=1}^{n}\beta_j \frac{\partial}{\partial y_j}\Big|_p$$

Then by using the result from $(*)$, we have —

$$v = \sum_{j=1}^{n}\beta_j\left(\sum_{i=1}^{n}\left(\frac{\partial}{\partial y_j}\Big|_p (x_i)\right)\frac{\partial}{\partial x_i}\Big|_p\right)$$

$$= \sum_{j=1}^{n}\sum_{i=1}^{n}\beta_j\left(\frac{\partial}{\partial y_j}\Big|_p (x_i)\right)\frac{\partial}{\partial x_i}\Big|_p$$

$$= \sum_{i=1}^{n}\left[\sum_{j=1}^{n}\beta_j\left(\frac{\partial}{\partial y_j}\Big|_p (x_i)\right)\right]\frac{\partial}{\partial x_i}\Big|_p$$

Therefore, $$\boxed{\alpha_i = \sum_{j=1}^{n}\beta_j\left(\frac{\partial}{\partial y_j}\Big|_p (x_i)\right)}$$

→ Also, for any tangent vector $v \in T_pM$, there exists at least one smooth curve $\gamma : (-\varepsilon, \varepsilon) \to M$ passing through $p \in M$ at $t=0$ and $v = \gamma'(0)$.

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BD | (7-6)

Let, (U, f) be a chart covering $p \in M$ and $(x_1, \ldots, x_m)$ be the associated coordinate map. Also, $f(p) = 0 \in \mathbb{R}^m$.

Then, any tangent vector v can be expressed as—

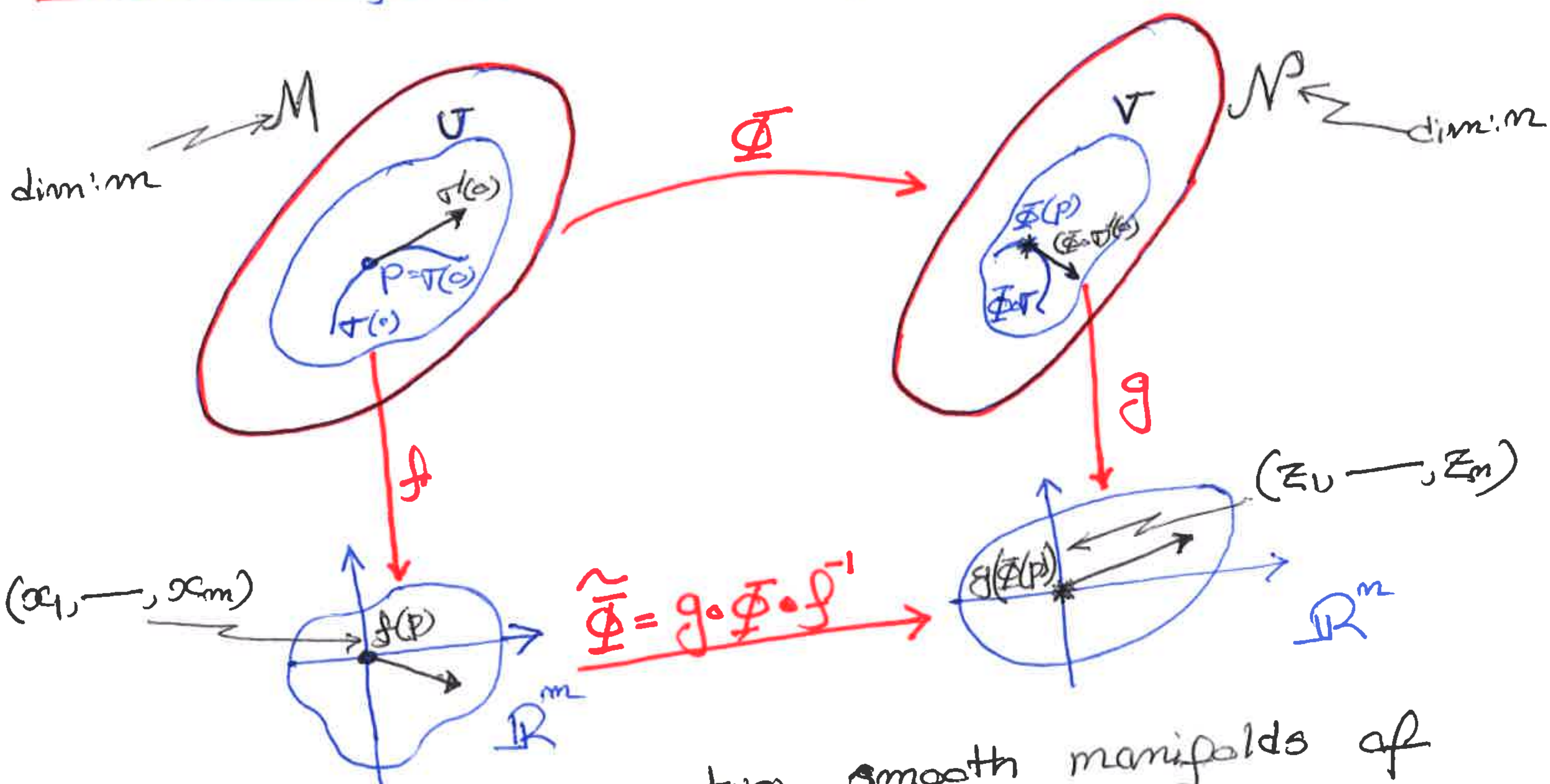
$$v = \sum_{i=1}^{m} \alpha_i \frac{\partial}{\partial x_i}\Big|_p \in T_pM$$

Now, we define the following curve—

$$\tau : (-\varepsilon, \varepsilon) \longrightarrow M$$
$$t \longmapsto f^{-1}(\alpha_1 t, \alpha_2 t, \ldots, \alpha_m t)$$

Then, it is easy to check that $\tau(0) = p$ and $\tau'(0) = v$.

Derivative of a smooth map between manifolds:—



→ Let, M and N be two smooth manifolds of dimension m and n, respectively, and $\Phi : M \to N$ is a smooth map (i.e. $\tilde{\Phi} \triangleq g\circ\Phi\circ f^{-1}$ is smooth for all appropriate chart (U, f) and (V, g)).

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→ Derivative of Φ at point $p \in M$ is a linear map from T_pM to $T_{\Phi(p)}N$. We denote it as Φ_{*p}.

→ First we consider the interpretation of tangent vectors as tangents to smooth curves. For a given $\omega \in T_pM$, we know that there is a smooth curve $\gamma : (-\epsilon, \epsilon) \to M$ such that $\gamma(0) = p \in M$ and $\gamma'(0) = \omega$. Then, $\Phi \circ \gamma : (-\epsilon, \epsilon) \to N$ gives us a smooth curve on N. Hence we can define Φ_{*p} as —

$$\Phi_{*p} : T_pM \longrightarrow T_{\Phi(p)}N$$
$$\omega = \gamma'(0) \longmapsto (\Phi \circ \gamma)'(0)$$

→ On the other hand, tangent vectors can also be perceived as derivations on the space of smooth functions, i.e. a tangent vector $\omega \in T_pM$ operates on smooth functions to give a real number at $p \in M$.

For a smooth function $\Psi : N \to \mathbb{R}$, the composition $\Psi \circ \Phi$ provides a real valued smooth function on M. This allows us to define the derivative $\Phi_{*p} : T_pM \to T_{\Phi(p)}N$ as —

$$\Phi_{*p}(\omega)(\Psi) = \omega(\Psi \circ \Phi)$$ for any $\Psi \in C^\infty(N)$.

→ It is worth mentioning here that $\Phi_{*p} : T_pM \to T_{\Phi(p)}N$ is also known as pushforward of $\Phi : M \to N$.

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→ $(\Phi\circ\gamma)'(0)(\Psi) = \frac{d}{dt}\left((\Psi\circ(\Phi\circ\gamma))(t)\right)\Big|_{t=0}$

$= \frac{d}{dt}\left(((\Psi\circ\Phi)\circ\gamma)(t)\right)\Big|_{t=0} = \gamma'(0)(\Psi\circ\Phi) = \omega(\Psi\circ\Phi)$

↑ Derivative of Φ from the tangent to a smooth curve perspective

↑ Derivative of Φ according to the derivation interpretation.

→ Now we assume $(x_1, \ldots, x_m)$ to be the coordinate map for the chart (U, f). Then $\omega \in T_pM$ can be expressed as —

$$\omega = \sum_{i=1}^{m} a_i \frac{\partial}{\partial x_i}\Big|_p$$

Similarly, $\Phi_{*p}(\omega) \in T_{\Phi(p)}N$ can be expressed as—

$$\Phi_{*p}(\omega) = \sum_{j=1}^{n} b_j \frac{\partial}{\partial z_j}\Big|_{\Phi(p)},$$

where $(z_1, \ldots, z_n)$ is the coordinate map for the chart (V, g).

Then,
$$\begin{bmatrix} b_1 \\ | \\ b_n \end{bmatrix} = \frac{\partial}{\partial x}\left(g\circ\Phi\circ f^{-1}\right)\Big|_{x=f(p)} \begin{bmatrix} a_1 \\ | \\ a_m \end{bmatrix}$$

↑ Local coord. representation of Φ_{*p}.

→ Let, $f:\mathbb{R}^n \to \mathbb{R}$ be a smooth function, and $M = \{x \in \mathbb{R}^n \mid f(x) = \text{constant}\}$. Then, for any $p \in M$, the tangent space T_pM is precisely the kernel of the derivative f_{*p}.

→ Some terminology:

— Φ_{*p} is one-to-one ⟶ Φ is an immersion

— Φ_{*p} is onto ⟶ Φ is a submersion

Tangent Bundle and Vector Field :—

10/10/2017
BD1 (7-8)

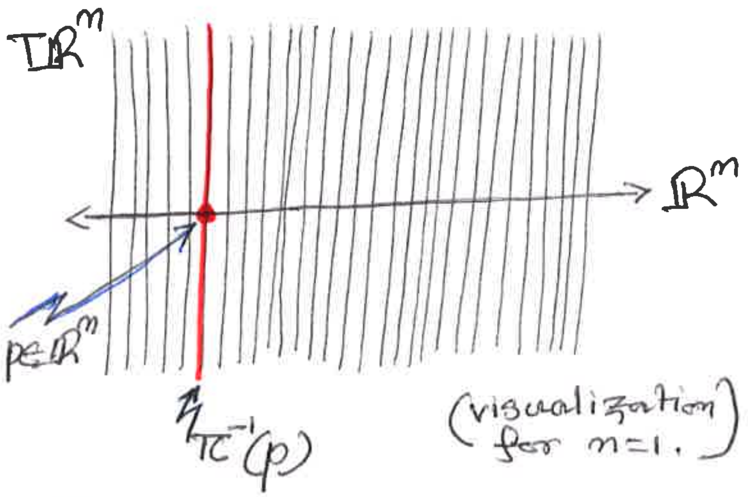
→ Tangent bundle : $TM = \bigcup_{p \in M} T_pM$

Then we can define a projection map :

$$\pi : TM \longrightarrow M$$
$$T_pM \ni v_p \longmapsto p$$

And, we call $\pi^{-1}(p)$ the fiber over $p \in M$. It is worth mentioning that the idea of projection and fiber is not constricted to tangent bundle only.

→ At any point $p \in \mathbb{R}^n$ we can assign a vector v, and denote it as the pair (p, v). The set of all such pairs is $\mathbb{R}^n \times \mathbb{R}^n$, which is same as $T\mathbb{R}^n$.



(visualization for $n=1$.)

→ In general, the tangent bundle TM of an n-dimensional manifold is a $2n$-dimensional manifold.

→ A smooth vector field X on a smooth manifold M is a smooth map —

$$X : M \longrightarrow TM$$
$$p \longmapsto (p, v_p)$$
($v_p \in T_pM$)

Clearly, $\pi \circ X : M \to M$ is the identity mapping $M \ni p \mapsto p \in M$.

Let (U, f) be a chart around $p \in M$ and $(x_1, \ldots, x_n)$ be the associated co-ord. map.

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Then we can write—

$$X(p) = \sum_{i=1}^{m} X_i(p) \frac{\partial}{\partial x_i}\Big|_p$$

— for brevity we often represent $(p, (X_1(p), \ldots, X_n(p))$ as $X(p) = (X_1(p), \ldots, X_n(p))$.

→ Consider $f, g_1, \ldots, g_m \in \mathfrak{X}(M)$ to be $(m+1)$ smooth vector fields on the n-dimensional smooth manifold M. $\mathfrak{X}(M)$ denotes the space of smooth vector fields on M.

Then, for all $x \in M$ we can define a dynamics as —

$$\dot{x} = f(x) + \sum_{i=1}^{m} u_i g_i(x),$$

where $(u_1, \ldots, u_m) \in U$, an open set in $\mathbb{R}^m$.

This defines an affine control system on M.

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