Lecture 8

MAE 544
NONLINEAR CONTROL

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(III) Example of a vector field on a sphere :-

Consider the 1-sphere S^1.

$$S^1 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = 1\} \subset \mathbb{R}^2$$

As we can perceive S^1 as a level set of the function —

$$f : \mathbb{R}^2 \to \mathbb{R}$$
$$(x_1, x_2) \mapsto x_1^2 + x_2^2,$$

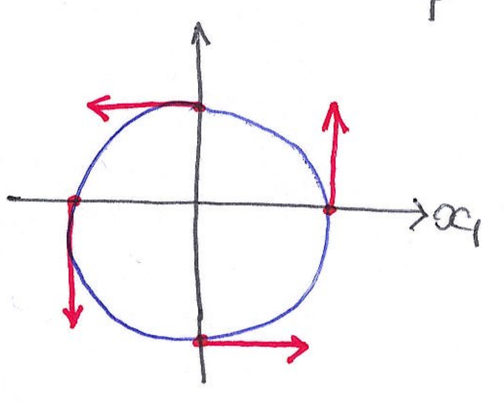
we can express the tangent space at $p = (x_1, x_2) \in S^1$ as —

$$T_p S^1 = \ker(f_{*p}) = \ker([x_1 \quad x_2])$$

Hence we can define a vector field on S^1 as —

$$X_p = -x_2 \frac{\partial}{\partial x_1}\Big|_p + x_1 \frac{\partial}{\partial x_2}\Big|_p \quad \text{— Smooth !!}$$

Note that X_p is nowhere zero as well !



→ Can we find a similar thing on S^2, i.e. is there a smooth vector field on S^2 which is nowhere zero?
— NO (Hedgehog / Hairy ball theorem)

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In general, we can always define a smooth vector field on S^n whenever n is an ~~even~~ odd number $(=2k+1, k \in \mathbb{N})$ such that the vector field is zero nowhere.

$$X_p = \left(-x_2 \frac{\partial}{\partial x_1}\Big|_p + x_1 \frac{\partial}{\partial x_2}\Big|_p\right) + \left(-x_4 \frac{\partial}{\partial x_3}\Big|_p + x_3 \frac{\partial}{\partial x_4}\Big|_p\right)$$
$$+ \cdots + \left(-x_{n+1} \frac{\partial}{\partial x_n}\Big|_p + x_n \frac{\partial}{\partial x_{n+1}}\Big|_p\right)$$

However, when n is an ~~odd~~ even number, such a vector field (which is smooth and zero nowhere) does not exist.

Derivative of a smooth real-valued function of M:—

Consider a real-valued smooth function f defined on a smooth manifold M.

$$f : M \to \mathbb{R}$$

Then, its derivative at point $p \in M$ (denoted as f_{*p} or $Df(p)$) is a mapping defined as —

$$f_{*p} : T_pM \longrightarrow T_{f(p)}\mathbb{R}$$
$$v \longmapsto f_{*p}(v)$$

Now, on $\mathbb{R}$ we can define a global chart by using a ~~cha~~ coordinate function —

$$x : \mathbb{R} \to \mathbb{R}$$
$$t \mapsto t$$

So, $\left\{ \frac{\partial}{\partial x}\Big|_q \right\}$ defines a basis for the tangent space $T_q\mathbb{R}$.

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Then, by using basis theorem, we have —

$$f_{*p}(v) = a \frac{\partial}{\partial x}\Big|_{f(p)}$$

where,

$$a = f_{*p}(v)(x) = v(x \circ f) = v(f).$$

Therefore, $$\boxed{f_{*p}(v) = v(f) \frac{\partial}{\partial x}\Big|_{f(p)}}$$

It is worth noting here that f_{*p} is also sometimes denoted as $df(p)$.

Then we can think of $df(p)$ as a linear functional on T_pM, i.e.

$$\boxed{\begin{aligned} df(p) : T_pM &\longrightarrow \mathbb{R} \\ v &\longmapsto v(f) \end{aligned}}$$

From this perspective we can say that —

$$df(p) \in T_p^*M,$$

the dual space of the tangent space at $p \in M$.

Cotangent Space:-

→ Cotangent space at point $p \in M$ is defined as the dual of the tangent space T_pM. Or in other words, cotangent space contains linear functionals which act on tangent vectors.

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→ Now, let (U, f) be a chart covering $p \in M$. And let $(x_1, \ldots, x_n)$ define the associated coordinate functions.

Then, $x_i : M \to \mathbb{R}$.

Similar to $f : M \to \mathbb{R}$, we can define $dx_i(p) \in T_p^* M$ as —

$$\boxed{dx_i(p)(v) = v(x_i)}$$

→ Earlier we have shown that $\left\{ \frac{\partial}{\partial x_i}\Big|_p \right\}_{i=1}^{n}$ defines a basis for T_pM.

$$dx_i(p)\left(\frac{\partial}{\partial x_j}\Big|_p\right) = \frac{\partial}{\partial x_j}\Big|_p (x_i) = \frac{\partial}{\partial \chi_j}\Big|_0 (x_i \circ f^{-1}) = \frac{\partial}{\partial \chi_j}\Big|_0 (\chi_i)$$

Therefore,

$$dx_i(p)\left(\frac{\partial}{\partial x_j}\Big|_p\right) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

Furthermore, it can be shown that $\{dx_i(p)\}_{i=1}^{n}$ is linearly independent, and any cotangent vector $\omega_p \in T_p^* M$ can be expressed as —

$$\boxed{\omega_p = \sum_{i=1}^{n} a_i \, dx_i(p),}$$

where,

$$\boxed{a_i = \omega_p\left(\frac{\partial}{\partial x_i}\Big|_p\right).}$$

Thus, $\{dx_i(p)\}_{i=1}^{m}$ provides a dual basis for the cotangent space T_p^*M.

For $v_p \in T_pM$ and $w_p \in T_p^*M$, they can be expressed as —

$$v_p = a_1 \frac{\partial}{\partial x_1}\Big|_p + \cdots + a_m \frac{\partial}{\partial x_m}\Big|_p = \begin{pmatrix} a_1 \\ | \\ a_m \end{pmatrix}$$

$$w_p = b_1\, dx_1(p) + \cdots + b_m\, dx_m(p) = \begin{pmatrix} b_1 \\ | \\ b_m \end{pmatrix}^T$$

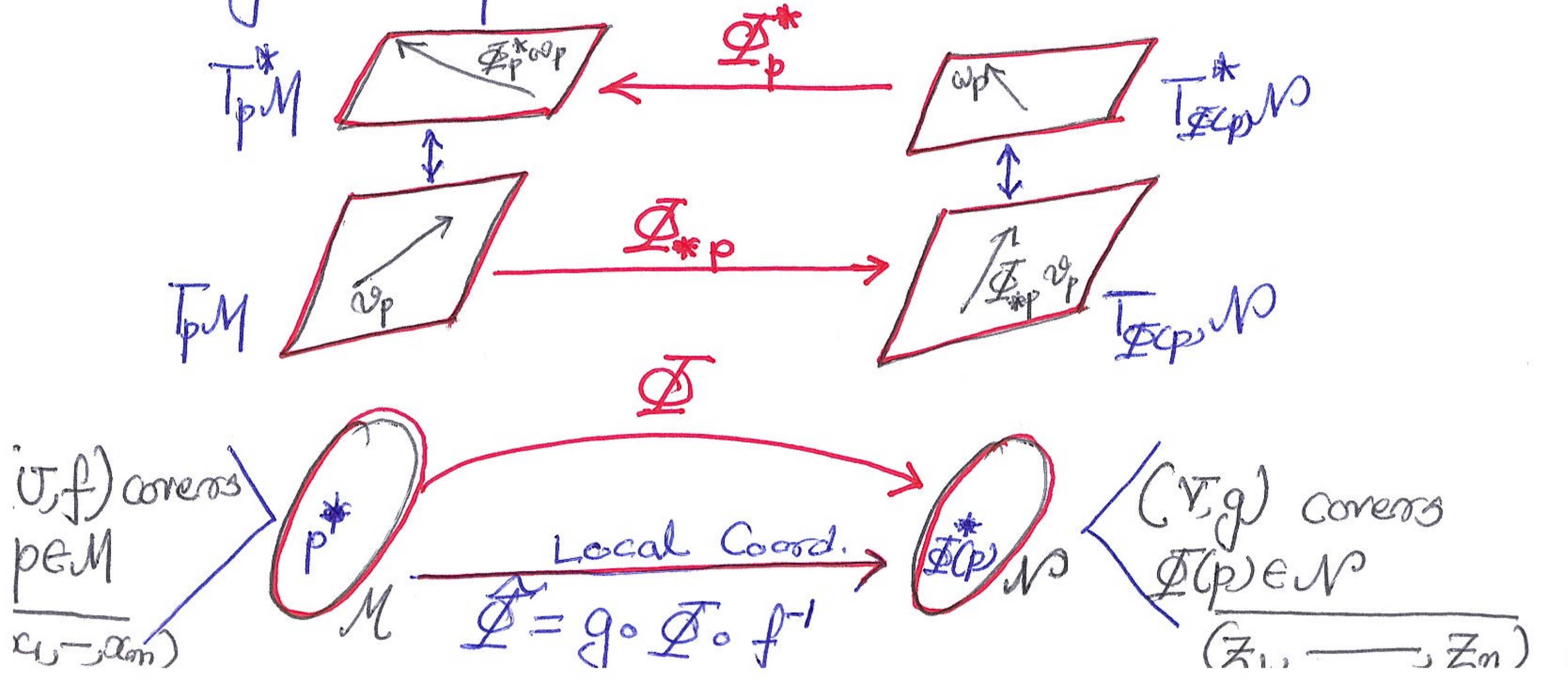
Coordinate Representation

Then,

$$w_p(v_p) = \sum_{i,j=1}^{m} a_i b_j\, dx_j(p)\left(\frac{\partial}{\partial x_i}\Big|_p\right)$$

$$= \sum_{i=1}^{m} a_i b_i$$ ← Local coordinates

Cotangent Maps :—



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We define the cotangent map as a linear mapping from $T^*_{\Phi(p)}N$ to T^*_pM, and denote it as Φ^*_p. It is also called the pullback of Φ at $p \in M$. Φ^*_p is adjoint to $\cancel{\Phi_{p*}}$ Φ_{*p}.

Now assume,

$$T_pM \ni v_p = a_1 \frac{\partial}{\partial x_1} + \text{———} + a_m \frac{\partial}{\partial x_m}$$

and, $$T_{\Phi(p)}N \ni \Phi_{*p}(v_p) = b_1 \frac{\partial}{\partial z_1} + \text{———} + b_m \frac{\partial}{\partial z_m}$$

and, $$T^*_{\Phi(p)}N \ni \omega_{\Phi(p)} = \beta_1\, dz_1 + \text{———} + \beta_m\, dz_m$$

As Φ^*_p is adjoint to Φ_{*p}, we have ———

$$\boxed{\omega_{\Phi(p)}\left(\Phi_{*p}(v_p)\right) = \Phi^*_p\,\omega_{\Phi(p)}\left(v_p\right)}$$

From previous discussion ———

$$\omega_{\Phi(p)}\left(\Phi_{*p}(v_p)\right) = \begin{bmatrix}\beta_1\\ |\\ \beta_m\end{bmatrix}^T \begin{bmatrix}b_1\\ |\\ b_m\end{bmatrix} = \begin{bmatrix}\beta_1\\ |\\ \beta_m\end{bmatrix}^T \left(\frac{\partial\tilde{\Phi}}{\partial x}\right)_{f(p)} \begin{bmatrix}a_1\\ |\\ a_m\end{bmatrix}$$

$$= \begin{bmatrix}\left(\frac{\partial\tilde{\Phi}}{\partial x}\right)^T_{f(p)} \begin{bmatrix}\beta_1\\ |\\ \beta_m\end{bmatrix}\end{bmatrix}^T \begin{bmatrix}a_1\\ |\\ a_m\end{bmatrix}$$

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Thus we have —

$$\left(\frac{\partial \tilde{\Phi}}{\partial x}\right)^T_{f(p)} \begin{pmatrix} \beta_1 \\ | \\ \beta_m \end{pmatrix}$$ as the local-coord. representation of $\Phi_p^* w_{\Phi(p)}$, and $\boxed{\left(\frac{\partial \Phi}{\partial x}\right)^T_{\Phi(p)}}$ as the local co-ord. representation of Φ_p^*.

Cotangent Bundle and Differential 1-forms :—

→ Cotangent Bundle : $\boxed{T^*M = \bigcup_{p \in M} T_p^* M}$

Then we can define a projection map (in similar way to what we did for tangent bundles) as —

$$\begin{aligned} \pi : T^*M &\longrightarrow M \\ T_p^*M \ni w_p &\longmapsto p \end{aligned}$$

→ A differential 1-form ω on this smooth manifold M is a smooth map —

$$\boxed{\omega : M \longrightarrow T^*M}$$
$$p \longmapsto (p, \omega_p)$$
$\omega_p \in T_p^*M$

Clearly, $\pi \circ \omega : M \to M$ is the identity mapping.

Also, similar to vector-fields, a differential 1-form can be expressed in the local coordinate as — $\boxed{\omega(p) = \sum_{i=1}^{n} \omega_i(p)\, dx_i(p).}$