Lecture 9

MAE 544

NONLINEAR CONTROL

10/17/2017

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- Some relevant aspects of vector fields:—

→ Let M be an n-dimensional smooth manifold. $C^\infty(M)$ and $\mathfrak{X}(M)$ denote the set of smooth real valued functions on M and the set of all smooth vector fields on M, respectively.

Now if we define the following operations—

i) $\mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$

$$(X+Y)(p) \triangleq X_p + Y_p = X(p) + Y(p) \in T_pM$$

for any $X, Y \in \mathfrak{X}(M)$ and $p \in M$

and, ii) $\mathbb{R} \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$

$$(\alpha X)(p) \triangleq \alpha X_p = \alpha X(p) \in T_pM \text{ for any } \alpha \in \mathbb{R}, X \in \mathfrak{X}(M)$$

Thus, $\mathfrak{X}(M)$ can be perceived as a vector space over $\mathbb{R}$.

→ Moreover, for any given $f \in C^\infty(M)$ we can define,

$$(fX)(p) \triangleq f(p)X_p \in T_pM \qquad \& \; X \in \mathfrak{X}(M)$$

As this defines ~~a mappi~~ a mapping

$$C^\infty(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M),$$

$\mathfrak{X}(M)$ can alternatively be viewed as a module over $C^\infty(M)$

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→ Another perspective towards vector fields.

Any vector field $X \in \mathfrak{X}(M)$ can also be viewed as the mapping $C^\infty(M) \to C^\infty(M)$, if we define

$$\boxed{\big(X(\Phi)\big)(p) \triangleq \underbrace{\big(X(p)\big)}_{T_pM}(\Phi) = X_p(\underbrace{\Phi}_{C^\infty(M)})} \quad p \in M, \Phi \in C^\infty(M)$$

According to this definition, for any two smooth functions $\Phi, \Psi \in C^\infty(M)$, we have—

$$\begin{aligned}\big(X(\Phi\Psi)\big)(p) &= X_p(\Phi\Psi) \quad X_p \in T_pM \\ &= \Phi(p)X_p(\Psi) + \Psi(p)X_p(\Phi) \\ &= \Phi(p)\big(X(\Psi)\big)(p) + \Psi(p)\big(X(\Phi)\big)(p) \\ &= \big(\Phi \cdot X(\Psi) + \Psi \cdot X(\Phi)\big)(p)\end{aligned}$$

Thus the mapping also satisfies the Leibnetz's rule, and hence, the set of vector fields <u>$\mathfrak{X}(M)$ can be perceived as the set of derivations over $C^\infty(M)$.</u>

⊛ In fact any linear map $Y: C^\infty(M) \to C^\infty(M)$ which satisfies the Leibnetz's rule, i.e. any derivation on the set of smooth real valued functions, can be induced by a vector field.

→ **Theorem:** If $\mathcal{D}$ is a derivation on $C^\infty(M)$, then there exists a unique vector field $X \in \mathfrak{X}(M)$ such that

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$$(\mathcal{D}\Phi)(p) = (X(p))(\Phi) \text{ for all } p \in M, \Phi \in C^\infty(M)$$

Proof: Our proof is carried out in three steps.

→ **Lemma 1:** If for ~~all~~ $X, Y \in \mathfrak{X}(M)$ and for all $\Phi \in C^\infty(M)$ we have $X(\Phi) = Y(\Phi)$, then $X = Y$.

Proof: As $X(\Phi) = Y(\Phi)$ implies $(X - Y)(\Phi) = 0$, it is sufficient if we can show that if $X(\Phi) = 0$ for all $\Phi \in C^\infty(M)$ then $X = 0$ (i.e., $X(p) = 0$ for all $p \in M$).

Let (U, f) be a chart covering $p \in M$, and let $(x_1, —, x_n)$ denote the associated coordinate functions.

Then $X \in \mathfrak{X}(M)$ can be expressed as $X = \sum_{i=1}^{n} X_i \frac{\partial}{\partial x_i}$ in the neighborhood U.

Also there exists a smooth function $\Psi \in C^\infty(M)$ such that Ψ is 1 in a neighborhood of p in U and Ψ is 0 outside U. Such a function is called a smooth bump function and its existence is ensured for a smooth manifold.

Then by setting $\Phi = \Psi \cdot x_i$ we have —

$$(X(\Phi))(p) = X(p)(\Psi x_i) = \sum_{j=1}^{n} X_j \frac{\partial}{\partial x_j}\Big|_p (\Psi \cdot x_i) = X_j(p)$$

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As $X(\Phi)=0$ for any $\Phi \in C^\infty(M)$ we have

$$X_j = 0 \text{ for all } j \in \{1, \dots, n\}.$$

Thus we have, $X=0$.

→ Lemma 2: If D is a derivation and $\Phi = 0$ on an open set U, then $D\Phi = 0$ on U.

Proof: To show this it is sufficient to show

$$D\Phi(p) = 0 \text{ for all } p \in U.$$

Now we can always find neighborhoods $V_p \subset U_p \subset U$ around p in U, such that we can define a function $\Psi \in C^\infty(M)$ such that Ψ is 1 on V_p and 0 outside U_p (i.e. on $M \setminus U_p$). Then, by introducing $\Theta \triangleq 1 - \Psi$, we can notice that

— whenever $q \in M \setminus U$, $(\Phi\Theta)(q) = \Phi(q) - \Phi(q)\underbrace{\Psi(q)}_{=0 \text{ as } M\setminus U \subset M \setminus U_p}$

$$= \Phi(q)$$

— whenever $q \in U$, $(\Phi\Theta)(q) = \Phi(q) - \Phi(q)\Psi(q)$

$$= 0 \longleftarrow \text{as } \Phi = 0 \text{ on } U$$

$$= \Phi(q)$$

Thus, we have $\boxed{\Phi = \Phi\Theta}$ on M.

Then,

$$D\Phi(p) = (D(\Phi\Theta))(p) = \boxed{\Phi(p)}\, D\Theta(p) + \boxed{\Theta(p)}\, D\Phi(p)$$

$$= 0$$

Both functions are zero at p.

This completes the proof for lemma 2.

→ **Final Step:**

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Let, $f \in C^\infty(M)$ be defined and smooth around p

Then we define, $\boxed{X_p(\Psi) = D\Phi(p)}$ for any $\Phi \in C^\infty(M)$ which agrees with Ψ around p. As D is a derivation, we have $X_p \in T_pM$.

Now consider, $\theta \in C^\infty(M)$ such that θ agrees with Ψ around p. Then we have $\Phi = \theta$ (i.e. $\Phi - \theta = 0$) on a neighborhood around p. Then from lemma 2, $D\Phi = D\theta$ on this neighborhood around p. Thus X_p is well defined.

In this way we can define a vector field $X : p \mapsto X_p$. It is yet to show that this is smooth, i.e. its coefficients in any local chart are smooth.

Let, $(x_1, \ldots, x_n)$ be the coordinate functions correspondi to a chart (U, f) covering $p \in M$.

Then $X = \sum_{i=1}^{n} X_i \frac{\partial}{\partial x_i}$ in ~~the~~ a neighborhood of p, where $X_i(p) = X_p(x_i) = Dx_i(p)$. As Dx_i is smooth for all $i \in \{1, \ldots, n\}$, X itself is smooth. □

→ $X : C^\infty(M) \to C^\infty(M)$, $\Phi \mapsto X(\Phi)$ such that $(X(\Phi))(p) = X_p(\Phi)$. $X(\Phi)$ is called the Lie-derivative of Φ with respect to X, and often denoted as $\mathcal{L}_X \Phi$.

III Control Systems as Vector fields and Integral curves:

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Let $X \in \mathfrak{X}(M)$. Then, as $X_p \in T_pM$, we know that there are many smooth curves $\Gamma : (-\varepsilon, \varepsilon) \to M$ such that $p = \Gamma(0)$ and $X_p = \Gamma'(0)$. However, whether there is a single smooth curve $\Gamma : (-\varepsilon, \varepsilon) \to M$ such that $\Gamma'(t) = X_{\Gamma(t)} = X(\Gamma(t))$ for any $t \in (-\varepsilon, \varepsilon)$, is yet to be explored.

→ A smooth curve $\Gamma : (-\varepsilon, \varepsilon) \to M$ is called an <u>integral curve</u> of $X \in \mathfrak{X}(M)$ passing through $p \in M$ if $\Gamma(0) = p$ and $\Gamma'(t) = X_{\Gamma(t)}$ for all $t \in (-\varepsilon, \varepsilon)$.

→ Using basis theorem we can express $X \in \mathfrak{X}(M)$ as—

$$X(\Gamma(t)) = \sum_{i=1}^{m} X(x_i) \left.\frac{\partial}{\partial x_i}\right|_{\Gamma(t)}$$

$(x_1, —, x_m)$ are the coordinate functions corresponding to a chart (U, f) covering $p \in M$.

On the other hand,

$$\Gamma'(t) = \sum_{i=1}^{m} \left(\left.\frac{d}{dt}(x_i \circ \Gamma)\right|_t \right) \left.\frac{\partial}{\partial x_i}\right|_{\Gamma(t)}.$$

Thus we have —

$$\boxed{\frac{d}{dt}(x_i \circ \Gamma) = X(x_i)}$$

$t \in (-\varepsilon, \varepsilon)$
$i \in \{1, —, m\}$
<u>initial value:</u> $x_i(p) = 0$
<u>Locality</u> $\{\Gamma(t)\}_{t \in (-\varepsilon, \varepsilon)} \subset U$

In local coordinates we can define $x \triangleq f \circ \gamma : (-\varepsilon, \varepsilon) \to f(U) \subseteq \mathbb{R}^n$ as a smooth curve on $\mathbb{R}^n$, and then we can think about integral curves as solution of the ODE —

$$\boxed{\frac{d}{dt}(x) = X(x) = X_x}$$

Solutions exist and they are unique because $X \in \mathfrak{X}(M)$

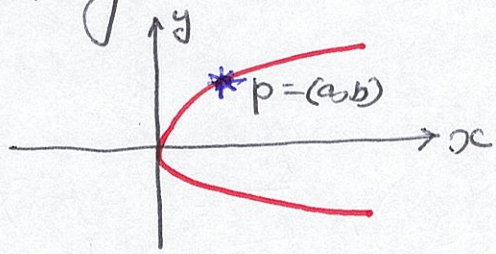
→ <u>Example</u>: $M = \mathbb{R}^2$, $X(x,y) = 2x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$, $p \in (a,b)$.

Then the associated ODEs are —

$$\frac{dx}{dt} = 2x \quad \text{with } x(0) = a \Rightarrow x(t) = e^{2t} \cdot a$$

<u>and</u>, $$\frac{dy}{dt} = y \quad \text{with } y(0) = b \Rightarrow y(t) = e^{t} b$$

The integral curve: $\gamma(t) = (ae^{2t}, be^{t})$



→ Let, $f, g_1, \ldots, g_m \in \mathfrak{X}(M)$ be $(m+1)$ smooth vector fields, then

$$\frac{d}{dt}(x) = f(x) + \sum_{i=1}^{m} u_i \, g_i(x), \quad x \in M$$
$$= \left(f + \sum_{i=1}^{m} g_i u_i\right)(x)$$

defines a <u>control system on M</u>. $\{u_i\}_{i=1}^{m}$ are called control inputs. This type of control systems are called <u>input affine system</u>, and $f(x), x \in M$ is called the <u>drift vector field</u>.

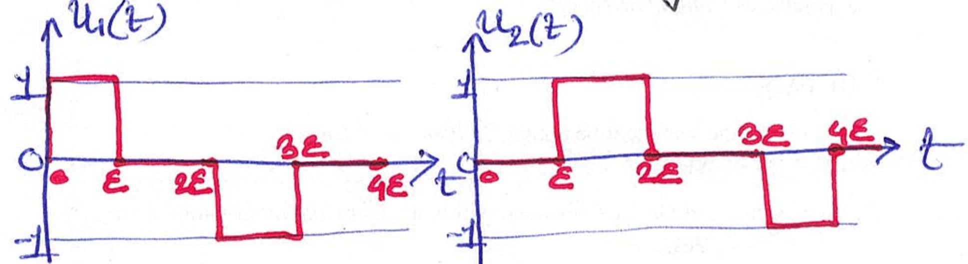
Lie Brackets in a Driftless Control System:- 10/17/2017 BD| (1-8)

Consider the following control system defined on $\mathcal{M} = \mathbb{R}^n$ —

$$\frac{d}{dt}\left(x(t)\right) = u_1(t)\, g_1\left(x_1(t)\right) + u_2(t)\, g_2\left(x(t)\right)$$

$x : (-T, T) \rightarrow \mathcal{M} = \mathbb{R}^n$

$g_1, g_2 \in \mathfrak{X}(\mathbb{R}^n)$



Then, by letting $\varepsilon \in (0, T/4)$ be sufficiently small, we have —

$$x(\varepsilon) = x_0 + \int_0^{\varepsilon} g_1\left(x(\tau_1)\right) d\tau_1 \qquad \left(\text{where } x(0) = x_0\right)$$

$$= x_0 + \int_0^{\varepsilon} g_1\left(x_0 + \int_0^{\tau_1} g_1\left(x(\tau_2)\right) d\tau_2\right) d\tau_1$$

$$= x_0 + \int_0^{\varepsilon}\left(g_1(x_0) + \frac{\partial g_1}{\partial x}(x_0) \cdot \left(\int_0^{\tau_1} g_1\left(x(\tau_2)\right) d\tau_2\right)\right) d\tau_1$$

$$= x_0 + \varepsilon \cdot g_1(x_0) + \frac{\partial g_1}{\partial x}(x_0) \int_0^{\varepsilon}\int_0^{\tau_1} g_1\left(x(\tau_2)\right) d\tau_2\, d\tau_1$$

$$= x_0 + \varepsilon g_1(x_0) + \frac{\partial g_1}{\partial x}(x_0) \int_0^{\varepsilon}\int_0^{\tau_1} g_1\left(x_0 + \int_0^{\tau_2} g_1\left(x(\tau_3)\right) d\tau_3\right) d\tau_2\, d\tau_1$$

$$= x_0 + \varepsilon g_1(x_0) + \frac{\partial g_1}{\partial x}(x_0) \int_0^{\varepsilon}\int_0^{\tau_1}\left(g_1(x_0) + \frac{\partial g_1}{\partial x}(x_0)\left(\int_0^{\tau_2} g_1\left(x(\tau_3)\right) d\tau_3\right)\right) d\tau_2\, d\tau_1$$

$$= x_0 + \varepsilon \cdot g_1(x_0) + \frac{\partial g_1}{\partial x}(x_0)\, g_1(x_0)\left(\int_0^{\varepsilon}\int_0^{\tau_1} d\tau_2\, d\tau_1\right)$$

$$+ \left(\frac{\partial g_1}{\partial x}(x_0)\right)^2 \int_0^{\varepsilon}\int_0^{\tau_1}\int_0^{\tau_2} g_1\left(x(\tau_3)\right) d\tau_3\, d\tau_2\, d\tau_1$$

Thus,

$x(\varepsilon)$

$$= x_0 + \varepsilon g_1(x_0) + \frac{\varepsilon^2}{2!}\frac{\partial g_1}{\partial x}(x_0)\, g_1(x_0) + O(\varepsilon^3)$$

In a similar way —

$x(2\varepsilon)$

$$= x(\varepsilon) + \varepsilon .\, g_2(x(\varepsilon)) + \frac{\varepsilon^2}{2!}\frac{\partial g_2}{\partial x}(x(\varepsilon))\, g_2(x(\varepsilon)) + O(\varepsilon^3)$$

$$= x_0 + \varepsilon g_1(x_0) + \frac{\varepsilon^2}{2!}\frac{\partial g_1}{\partial x}\Big|_{x_0} g_1(x_0) + \varepsilon g_2(x_0) + \varepsilon^2 \frac{\partial g_2}{\partial x}\Big|_{x_0} g_1(x_0)$$
$$+ \frac{\varepsilon^2}{2!}\frac{\partial g_2}{\partial x}\Big|_{x_0} g_2(x_0) + O(\varepsilon^3)$$

$$= x_0 + \varepsilon .\left(g_1(x_0) + g_2(x_0)\right) + \frac{\varepsilon^2}{2!}\left(\frac{\partial g_1}{\partial x}\Big|_{x_0} g_1(x_0) + \frac{\partial g_2}{\partial x}\Big|_{x_0} g_2(x_0)\right.$$
$$\left. + 2\frac{\partial g_2}{\partial x}\Big|_{x_0} g_1(x_0)\right) + O(\varepsilon^3)$$

Then we have —

$x(3\varepsilon)$

$$= x(2\varepsilon) - \varepsilon g_1(x(2\varepsilon)) + \frac{\varepsilon^2}{2!}\frac{\partial g_1}{\partial x}(x(2\varepsilon))\, g_1(x(2\varepsilon)) + O(\varepsilon^3)$$

$$= x_0 + \varepsilon .\left(g_1(x_0) + g_2(x_0)\right) + \frac{\varepsilon^2}{2!}\left(\frac{\partial g_1}{\partial x}\Big|_{x_0} g_1(x_0) + \frac{\partial g_2}{\partial x}\Big|_{x_0} g_2(x_0) + 2\frac{\partial g_2}{\partial x}\Big|_{x_0} g_1(x_0)\right)$$
$$- \varepsilon .\, g_1(x_0) - \varepsilon^2 \frac{\partial g_1}{\partial x}\Big|_{x_0}\left(g_1(x_0) + g_2(x_0)\right) + \frac{\varepsilon^2}{2!}\frac{\partial g_1}{\partial x}\Big|_{x_0} g_1(x_0) + O(\varepsilon^3)$$

$$= x_0 + \varepsilon .\, g_2(x_0) + \frac{\varepsilon^2}{2!}\left(\frac{\partial g_2}{\partial x}\Big|_{x_0} g_2(x_0) + 2 .\frac{\partial g_2}{\partial x}\Big|_{x_0} g_1(x_0)\right.$$
$$\left. - 2\frac{\partial g_1}{\partial x}\Big|_{x_0} g_2(x_0)\right) + O(\varepsilon^3)$$

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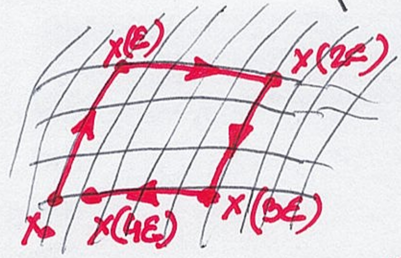
Finally,

$$
\begin{aligned}
x(4\varepsilon) &= x(3\varepsilon) - \varepsilon g_2(x(3\varepsilon)) + \frac{\varepsilon^2}{2!}\frac{\partial g_2}{\partial x}(x(3\varepsilon))\, g_2(x(3\varepsilon)) + O(\varepsilon^3) \\
&= x_0 + \varepsilon g_2(x_0) + \frac{\varepsilon^2}{2!}\left(\frac{\partial g_2}{\partial x}\Big|_{x_0} g_2(x_0) + 2\left(\frac{\partial g_2}{\partial x}\Big|_{x_0} g_1(x_0) - \frac{\partial g_1}{\partial x}\Big|_{x_0} g_2(x_0)\right)\right) \\
&\quad - \varepsilon\, g_2(x_0) - \varepsilon^2 \frac{\partial g_2}{\partial x}\Big|_{x_0} g_2(x_0) + \frac{\varepsilon^2}{2!}\frac{\partial g_2}{\partial x}\Big|_{x_0} g_2(x_0) + O(\varepsilon^3) \\
&= x_0 + \varepsilon^2 \underbrace{\left(\frac{\partial g_2}{\partial x}\Big|_{x_0} g_1(x_0) - \frac{\partial g_1}{\partial x}\Big|_{x_0} g_2(x_0)\right)}_{\triangleq\, [g_1,\, g_2](x_0)} + O(\varepsilon^3)
\end{aligned}
$$



$\triangleq [g_1, g_2](x_0)$ ← Lie-bracket

- It can give rise to motion which were not possible with individual vector fields alone.

→ Now, $[\cdot, \cdot] : \mathfrak{X}(\mathbb{R}^n) \times \mathfrak{X}(\mathbb{R}^n) \to \mathfrak{X}(\mathbb{R}^n)$ defines a second binary operation on $\mathfrak{X}(\mathbb{R}^n)$, which we have shown earlier is a vector space over $\mathbb{R}$. In fact, with this Lie-bracket operation, we can show that $\mathfrak{X}(\mathbb{R}^n)$ is an algebra.

Moreover,

— $[g_1, g_2] = -[g_1, g_2]$

— $[g_1, [g_2, g_3]] + [g_2, [g_3, g_1]] + [g_3, [g_1, g_2]] = 0$

for any $g_1, g_2, g_3 \in \mathfrak{X}(\mathbb{R}^n)$

Thus, $\mathfrak{X}(\mathbb{R}^n)$ is a Lie-algebra.

An Example:– (Non-holonomic Integrator)

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$$\boxed{\begin{aligned}\dot{x} &= u_1\\ \dot{y} &= u_2\\ \dot{z} &= x u_2 - y u_1\end{aligned}} \Bigg\} \Rightarrow \frac{d}{dt}\begin{pmatrix}x\\y\\z\end{pmatrix} = \underbrace{\begin{pmatrix}1\\0\\-y\end{pmatrix}}_{g_1} u_1 + \underbrace{\begin{pmatrix}0\\1\\x\end{pmatrix}}_{g_2} u_2$$

$$\Downarrow$$

$$\dot{x}y - \dot{y}x + \dot{z} = 0$$

$$\Rightarrow \begin{pmatrix}y\\-x\\1\end{pmatrix}^T \begin{pmatrix}\dot{x}\\\dot{y}\\\dot{z}\end{pmatrix} = 0 \longrightarrow$$ gives us a constraint on the possible velocities; or, in other words, only a subset of the tangent space can be achieved.

Now, $$[g_1, g_2] = \frac{\partial g_2}{\partial x} g_1 - \frac{\partial g_1}{\partial x} g_2 = \begin{bmatrix}0&0&0\\0&0&0\\1&0&0\end{bmatrix}\begin{bmatrix}1\\0\\-y\end{bmatrix} - \begin{bmatrix}0&0&0\\0&0&0\\0&-1&0\end{bmatrix}\begin{bmatrix}0\\1\\x\end{bmatrix}$$

$x = \begin{pmatrix}x\\y\\z\end{pmatrix}$

$$= \begin{bmatrix}0\\0\\2\end{bmatrix}$$

Note that g_1, g_2 and $[g_1, g_2]$ are all linearly independent. As a consequence, by switching between the control inputs we can move the system state in certain directions which were not initially possible.

Fun fact:

$$u_1 = -\alpha A \sin(\alpha t + \phi)$$
$$u_2 = \alpha A \cos(\alpha t + \phi)$$

$$z = \alpha A^2 t$$

$$\uparrow$$

Then, $$\left[\begin{aligned} x &= A\cos(\alpha t + \phi)\\ y &= A\sin(\alpha t + \phi)\end{aligned}\right\} \rightarrow \dot{z} = \alpha A^2$$