Lecture 10

MAE 544
NONLINEAR CONTROL

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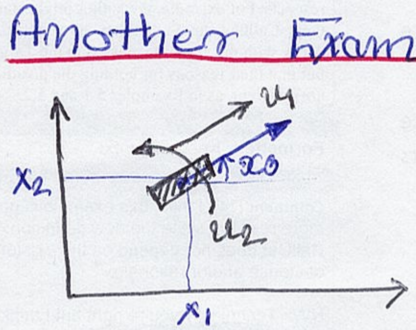
Another Example: Unicycle



Position: (x_1, x_2)

Orientation: x_3

$\hookrightarrow$ system state: (x_1, x_2, x_3)

Speed: u_1

Turning Rate: u_2

$\hookrightarrow$ control inputs: (u_1, u_2)

Dynamics

$$\left.\begin{aligned} \dot{x}_1 &= u_1 \cdot \cos x_3 \\ \dot{x}_2 &= u_1 \sin x_3 \\ \dot{x}_3 &= u_2 \end{aligned}\right\} \Longrightarrow \frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \underbrace{\begin{pmatrix} \cos x_3 \\ \sin x_3 \\ 0 \end{pmatrix}}_{= g_1(x)} u_1 + \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{= g_2(x)} u_2$$

As vector fields: —

$$g_1 = \cos x_3 \frac{\partial}{\partial x_1} + \sin x_3 \frac{\partial}{\partial x_2} \longleftarrow \text{drive}$$

$$g_2 = \frac{\partial}{\partial x_3} \longleftarrow \text{rotate.}$$

Constraints: $\boxed{(\sin x_3)\,\dot{x}_1 - (\cos x_3)\,\dot{x}_2 = 0}$ $\longrightarrow$ No lateral motion.

$\rightarrow$ It is a nonholonomic constraint.

Now,

$$[g_1, g_2] = \underbrace{\frac{\partial g_2}{\partial x}}_{=0} g_1 - \frac{\partial g_1}{\partial x} g_2 = -\begin{bmatrix} 0 & 0 & -\sin x_3 \\ 0 & 0 & \cos x_3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \sin x_3 \\ -\cos x_3 \\ 0 \end{bmatrix}$$

→ Note that g_1, g_2 and $[g_1, g_2]$ are linearly independent, and at any point $x = (x_1, x_2, x_3)$ in the state space $g_1(x)$, $g_2(x)$ and $[g_1, g_2](x)$ defines a basis for the associated tangent space. Moreover, $[g_1, g_2]$ is orthogonal to the drive vector field (i.e. g_1).

Lie-Bracket of vector fields:-

Let, $X, Y \in \mathfrak{X}(M)$ and $\Phi \in C^\infty(M)$ where M is an n-dimensional smooth manifold. As discussed in our last lecture $\mathfrak{X}(M)$ can be identified with space of derivations over M.

$$[X, Y](\Phi) \triangleq X(Y(\Phi)) - Y(X(\Phi))$$

$$= [L_X L_Y - L_Y L_X](\Phi)$$

→ In Local coordinates:

If $X = \sum_{i=1}^{n} X_i \frac{\partial}{\partial x_i}$ and $Y = \sum_{i=1}^{n} Y_i \frac{\partial}{\partial x_i}$,

$[X, Y](\Phi)$

$$= \sum_{i=1}^{n} X_i \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{n} Y_j \frac{\partial \Phi}{\partial x_j}\right) - \sum_{i=1}^{n} Y_i \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{n} X_j \frac{\partial \Phi}{\partial x_j}\right)$$

$$= \sum_{i,j=1}^{n} \left(X_i \frac{\partial}{\partial x_i}\left(Y_j \frac{\partial \Phi}{\partial x_j}\right) - Y_i \frac{\partial}{\partial x_i}\left(X_j \frac{\partial \Phi}{\partial x_j}\right)\right)$$

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$$= \sum_{i,j=1}^{m} \left(X_i \frac{\partial Y_j}{\partial x_i} \frac{\partial \Phi}{\partial x_j} + X_i Y_j \cancel{\frac{\partial^2 \Phi}{\partial x_i \partial x_j}} - Y_i \frac{\partial X_j}{\partial x_i} \frac{\partial \Phi}{\partial x_j} - X_j Y_i \cancel{\frac{\partial^2 \Phi}{\partial x_i \partial x_j}} \right)$$

$$= \sum_{j=1}^{m} \left(\sum_{i=1}^{m} \left(\frac{\partial Y_j}{\partial x_i} X_i - \frac{\partial X_j}{\partial x_i} Y_i \right) \right) \frac{\partial \Phi}{\partial x_j}$$

$$= \left(\frac{\partial Y}{\partial x} X - \frac{\partial X}{\partial x} Y \right)(\Phi)$$

Thus, $\boxed{[X, Y](\Phi) = \left(\frac{\partial Y}{\partial x} X - \frac{\partial X}{\partial x} Y \right)(\Phi)}$

Clearly, $[X, Y] = -[Y, X]$, and $[X, X] = 0$.

→ If $X, Y \in \mathfrak{X}(M)$ and $\Phi, \Psi \in C^\infty(M)$, then $\Phi X, \Psi Y \in \mathfrak{X}(M)$ as well.

$[\Phi X, \Psi Y](\Theta) \qquad \Theta \in C^\infty(M)$

$$= (\Phi X)\big((\Psi Y)(\Theta)\big) - (\Psi Y)\big((\Phi X)(\Theta)\big)$$

$$= \Phi X\big(\Psi \cdot Y(\Theta)\big) - \Psi Y\big(\Phi \cdot X(\Theta)\big)$$

$$= \Phi \Psi X(Y(\Theta)) + \Phi X(\Psi) Y(\Theta) - \Psi Y(\Phi) X(\Theta) - \Psi \Phi Y(X(\Theta))$$

$$= \Phi \Psi \big(X(Y(\Theta)) - Y(X(\Theta)) \big) + \big(\Phi X(\Psi) Y - \Psi Y(\Phi) X \big)(\Theta)$$

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Thus, $\boxed{[\Phi X, \Psi Y] = \Phi\Psi[X,Y] + \Phi X(\Psi)Y - \Psi Y(\Phi)X}$

→ Let $X, Y, Z \in \mathfrak{X}(M)$. Then for any $\Phi \in C^\infty(M)$ —

$$[X,[Y,Z]](\Phi)$$

$$= X\big([Y,Z](\Phi)\big) - [Y,Z]\big(X(\Phi)\big)$$

$$= X\Big(Y\big(Z(\Phi)\big) - Z\big(Y(\Phi)\big)\Big) - \Big(Y\big(Z(X(\Phi))\big) - Z\big(Y(X(\Phi))\big)\Big)$$

$$= L_X\big(L_Y(L_Z\Phi)\big) - L_X\big(L_Z(L_Y\Phi)\big) - L_Y\big(L_Z(L_X\Phi)\big) + L_Z\big(L_Y(L_X\Phi)\big)$$

$$= \big[L_X\circ L_Y\circ L_Z + L_Z\circ L_Y\circ L_X - L_X\circ L_Z\circ L_Y - L_Y\circ L_Z\circ L_X\big](\Phi)$$

In a similar way

$$[Y,[Z,X]](\Phi)$$

$$= \big[L_Y\circ L_Z\circ L_X + L_X\circ L_Z\circ L_Y - L_Y\circ L_X\circ L_Z - L_Z\circ L_X\circ L_Y\big](\Phi)$$

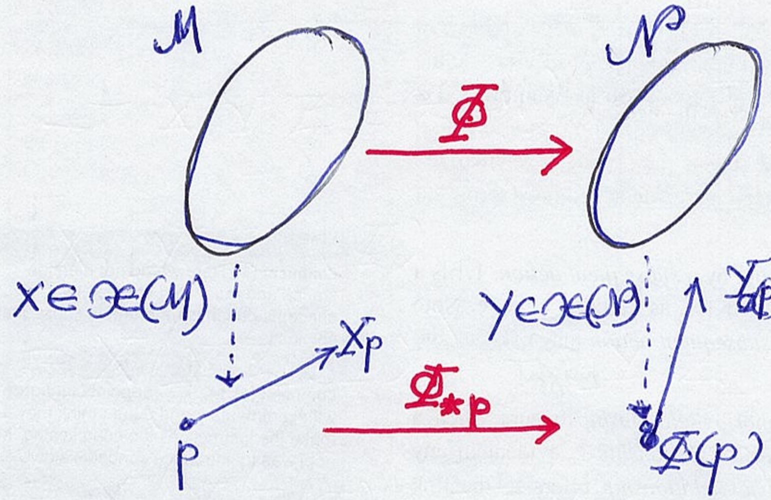
$$[Z,[X,Y]](\Phi)$$

$$= \big[L_Z\circ L_X\circ L_Y + L_Y\circ L_X\circ L_Z - L_Z\circ L_Y\circ L_X - L_X\circ L_Y\circ L_Z\big](\Phi)$$

Then it readily follows —

$\boxed{[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0}$ ← Jacobi Identity

→ Push-Forward:



- $X \in \mathfrak{X}(\mathcal{M})$ and $Y \in \mathfrak{X}(\mathcal{M})$ are Φ-related if —

  $$Y_{\Phi(p)} = \Phi_{*p} X_p$$ for all $p \in \mathcal{M}$.

- If $\Phi : \mathcal{M} \to \mathcal{N}$ is a diffeomorphism, we define its pushforward $\Phi_* : T\mathcal{M} \to T\mathcal{N}$ as —

$$\boxed{(\Phi_* X)_q = \Phi_{*\Phi^{-1}(q)} X_{\Phi^{-1}(q)}} \qquad q \in \mathcal{N}$$

- If $X_1, X_2 \in \mathfrak{X}(\mathcal{M})$ and $Y_1, Y_2 \in \mathfrak{X}(\mathcal{N})$, then we define the pushforward of the Lie-bracket as —

$$\boxed{\Phi_*[X_1, X_2] = [\Phi_* X_1, \Phi_* X_2]} = [Y_1, Y_2]$$

Let, $g \in C^\infty(\mathcal{N})$.

Then, $g \circ \Phi \in C^\infty(\mathcal{M})$, and, $\Phi_* X(g) = X(g \circ \Phi)$.

also denoted by $\Phi^* g$

Therefore,

$$[X_1, X_2](g \circ \Phi)(p) = [\Phi_* X_1, \Phi_* X_2](g)(\Phi(p))$$

$$\begin{tikzcd}
\mathfrak{X}(\mathcal{M}) \arrow[r, "\Phi^*"] \arrow[d, leftrightarrow] \arrow[dr, phantom, "="] & \mathfrak{X}(\mathcal{N}) \arrow[d, leftrightarrow] \\
C^\infty(\mathcal{M}) & C^\infty(\mathcal{N}) \arrow[l, "\Phi^*"]
\end{tikzcd}$$

(III) Lie-Algebra:

→ A real vector space V is a Lie-algebra if in addition to its linear structure there also exists a second binary operation $[\cdot,\cdot] : V \times V \to V$ such that the following holds true for all $v_1, v_2, v_3 \in V$ and $\alpha, \beta \in \mathbb{R}$ —

i) $[\alpha v_1 + \beta v_2, v_3] = \alpha[v_1, v_3] + \beta[v_2, v_3]$ ← Bilinearity

ii) $[v_1, v_2] = -[v_2, v_1]$ ← skew-symmetry

iii) $[v_1,[v_2,v_3]] + [v_2,[v_3,v_1]] + [v_3,[v_1,v_2]] = 0$ ← Jacobi Identity

→ A subalgebra $(\tilde{V})$ of a Lie-algebra $(V, [\cdot,\cdot])$ is a linear subspace $\tilde{V} \subset V$ such that $[\tilde{v}, \tilde{w}] \in \tilde{V}$ for all $\tilde{v}, \tilde{w} \in \tilde{V}$ (i.e. $(\tilde{V}, [\cdot,\cdot])$ is a Lie-algebra on its own).

→ Some examples:

→ $V = \mathfrak{X}(M)$, $[\cdot,\cdot] \to$ Lie-bracket of vector fields

→ $V = gl(n)$, $[\cdot,\cdot] : gl(n) \times gl(n) \to gl(n)$, $A, B \mapsto (AB - BA)$

$gl(n)$ ← space of all $n \times n$ real matrices, i.e. $\mathbb{R}^{n\times n}$.

→ $V = so(n) = \{A \in gl(n) = \mathbb{R}^{n\times n} \mid \underbrace{A^T + A}_{=0}\}$ ← space of $n\times n$ skew-symm mat.

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so(n) is a subalgebra of gl(n) with

$[\cdot,\cdot] : A, B \longmapsto (AB - BA)$.

$$(AB-BA)^T + (AB - BA)$$
$$\left.\begin{aligned} &= (AB)^T - (BA)^T + AB - BA \\ &= B^T A^T - A^T B^T + AB - BA \\ &= BA - AB + AB - BA \end{aligned}\right\} (AB - BA) \in so(n)$$
$$= 0$$

$\rightarrow$ $V = se(n) = \left\{ A \in gl(n+1) \,\middle|\, A = \begin{bmatrix} \Omega & v \\ 0 & 0 \end{bmatrix}, \Omega \in so(n), v \in \mathbb{R}^n \right\}$

se(n) is a subalgebra of gl(n+1) with the binary operation $[\cdot,\cdot] : A, B \longmapsto AB - BA$.

Let, $A = \begin{bmatrix} \Omega_1 & v_1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} \Omega_2 & v_2 \\ 0 & 0 \end{bmatrix}$, $\Omega_1, \Omega_2 \in so(n)$
$v_1, v_2 \in \mathbb{R}^n$

Then,

$$AB - BA = \begin{bmatrix} \Omega_1 & v_1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \Omega_2 & v_2 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} \Omega_2 & v_2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \Omega_1 & v_1 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \Omega_1\Omega_2 - \Omega_2\Omega_1 & \Omega_1 v_2 - \Omega_2 v_1 \\ 0 & 0 \end{bmatrix} \in se(n)$$

$\rightarrow$ Structure constants:—

Let, $(B_1, \ldots, B_m)$ be a basis for the Lie-algebra V.
Then, we can write,

$$[B_i, B_j] = \sum_{k=1}^{m} \boxed{\Gamma_{ij}^{k}}\, B_k$$

structure constants

Lie-Groups:

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→ A Lie Group G is a smooth manifold which is also a group, such that the group operations of multiplication and inverse are smooth, i.e.

$$\Psi : G \times G \longrightarrow G$$
$$(g_1, g_2) \longmapsto g_1 \cdot g_2^{-1}$$

is a smooth map.

→ Some examples:

→ $G = \mathbb{R}^n$ with vector addition

→ $G = GL(n) = \{ g \in \mathbb{R}^{n \times n} \mid \det(g) \neq 0 \}$ under matrix

→ $G = SO(n) = \{ g \in GL(n) \subset \mathbb{R}^{n \times n} \mid g^T g = I, \det(g) = 1 \}$ under matrix multiplication.

→ $G = SE(n) = \left\{ g = \begin{bmatrix} R & b \\ 0 & 1 \end{bmatrix} \in GL(n+1) \,\middle|\, R \in SO(n), b \in \mathbb{R}^n \right\}$

→ Let, G and $\tilde{G}$ be two Lie-groups. Then —

$G \times \tilde{G} \triangleq \{ (g, \tilde{g}) \mid g \in G, \tilde{g} \in \tilde{G} \}$ is a Lie-group under the group operation —

$$(g, \tilde{g}) \cdot (h, \tilde{h}) = (g \cdot h, \tilde{g} \cdot \tilde{h})$$

→ Left and Right Translation:

- Left Translation: $L_g : G \longrightarrow G$, $h \longmapsto g \cdot h$
- Right Translation: $R_g : G \longrightarrow G$

Then following holds true —

$\rightarrow L_{g_1} \circ L_{g_2} = L_{g_1 \cdot g_2}$

$\rightarrow (L_g)^{-1} = L_{g^{-1}}$

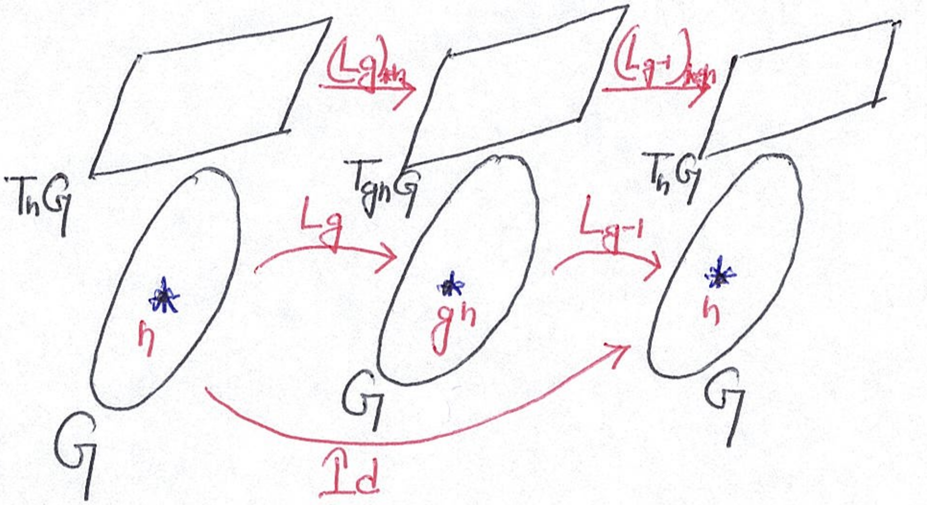
$\rightarrow R_{g_1} \circ R_{g_2} = R_{g_2 \cdot g_1}$

$\rightarrow (R_g)^{-1} = R_{g^{-1}}$

$\rightarrow L_e = \mathrm{Id.} = R_e$ where $e \in G$ is the group identity element.

$\rightarrow \mathrm{Id} = (L_{g^{-1}} \circ L_g)_{*h} = (L_{g^{-1}})_{*gh} \circ (L_g)_{*h}$

$(L_g)_{*h}$ is invertible.



→ <u>Charts</u>:—

Let (U, f) be a chart covering $e \in G$.

Then around any point $g \in G$, we can define a chart (U_g, f_g) such that —

$$U_g = L_g(U) = \{gh \mid h \in U\}$$

$$f_g = f \circ L_{g^{-1}} = L^*_{g^{-1}} f : U_g \longrightarrow \mathbb{R}^n$$