Lecture 11

MAE 544
NONLINEAR CONTROL

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Revisiting Unicycle:—

$$\left.\begin{aligned}\dot{x}_1 &= u_1 \cos x_3\\ \dot{x}_2 &= u_1 \sin x_3\\ \dot{x}_3 &= u_2\end{aligned}\right\} \Leftrightarrow \frac{d}{dt}\left[\begin{array}{cc|c}\cos x_3 & -\sin x_3 & x_1\\ \sin x_3 & \cos x_3 & x_2\\ \hline 0 & 0 & 1\end{array}\right]$$

$$= \underbrace{\left[\begin{array}{cc|c}\cos x_3 & -\sin x_3 & x_1\\ \sin x_3 & \cos x_3 & x_2\\ \hline 0 & 0 & 1\end{array}\right]}_{= g \,\in\, SE(2)} \underbrace{\left[\begin{array}{cc|c}0 & -u_2 & u_1\\ u_2 & 0 & 0\\ \hline 0 & 0 & 0\end{array}\right]}_{= \xi_u \,\in\, se(2)}$$

$$\boxed{\dot{g} = g\,\xi_u} \Leftrightarrow$$

→ Unicycle dynamics can be perceived as a family of vector fields on $SE(2)$, parametrized by control inputs u_1 and u_2.

→ At the group identity element (i.e. at $g = \mathbb{I}_3$) the tangent space can be identified with a subspace of $se(2)$. Also, the vector field is well defined, once we have defined the tangent vectors at the identity element of the underlying Lie group $SE(2)$.

Left-Invariant Vector Fields:

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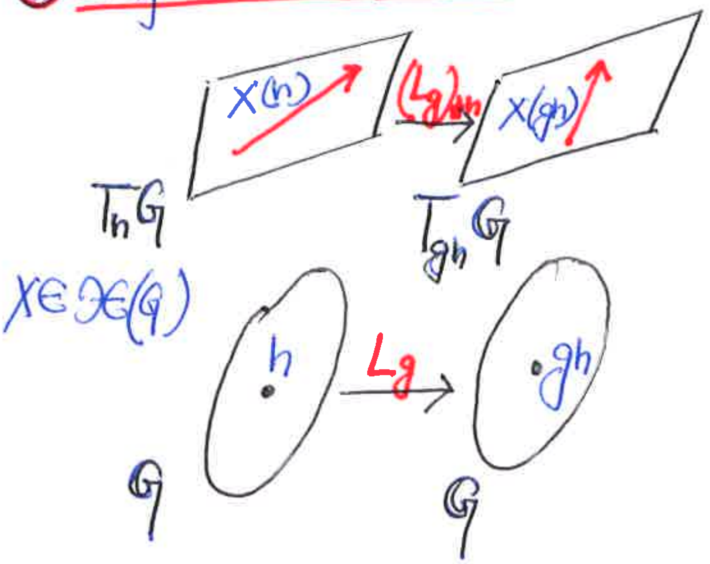

→ A smooth vector field $X\in\mathfrak{X}(G)$ is called <u>left-invariant</u> if —

$$X(L_g h) = (L_g)_* X(h)$$

for every $g, h\in G$. In other words X is a left invariant vector field if it is <u>L_g-related to itself</u>.

$\mathfrak{X}(G)$: Space of smooth vector fields on G

$\mathfrak{X}_L(G)$: Space of left-invariant vector fields on G.

→ For $X, Y\in\mathfrak{X}_L(G)$ and $g\in G$, we have —

$$\cancel{(L_g)_{*h}[X,Y](h) = [(L_g)_{*h}X, (L_g)_{*h}Y](h)}\quad \cancel{(L_g)_{*h}[X(h), Y(h)]}$$

$$\cancel{= [(L_g)_{*h}X(h), (L_g)_{*h}Y(h)]}$$

$$\cancel{= [X(L_g h), Y(L}$$

$$\left((L_g)_{*h}[X,Y](h)\right)(\Phi) = \left([(L_g)_{*h}X, (L_g)_{*h}Y](\Phi)\right)(L_g h) \qquad \Phi\in C^\infty(G),\ h\in G$$

$$= \left([X,Y](\Phi)\right)(gh)$$

$$= \left([X,Y](L_g h)\right)(\Phi)$$

Hence, $[X, Y] \in \mathfrak{X}_L(G)$, and thus $\mathfrak{X}_L(G)$ is a lie-subalgebra of $\mathfrak{X}(G)$.

→ The lie-algebra of left-invariant vector fields on a Lie-group is called the lie-algebra of the lie-group.

→ Let, $X \in \mathfrak{X}_L(G)$ be a left-invariant vector field on G. Then, $X(g) = (L_g)_{*e} X(e)$ where $e \in G$ is the identity element of G. Thus a left-invariant vector field is completely defined by its evaluation at $e \in G$, i.e. $X(e) \in T_e G$.

On the other hand, for any $\xi \in T_e G$, we can define a vector field as —

$$X_\xi(g) = (L_g)_{*e}\, \xi.$$

Clearly —

$$\begin{aligned} X_\xi(L_g h) = X_\xi(gh) &= (L_{gh})_{*e}\, \xi \\ &= (L_g \circ L_h)_{*e}\, \xi \\ &= (L_g)_{*h} (L_h)_{*e}\, \xi \\ &= (L_g)_{*h}\, X_\xi(h). \end{aligned}$$

Thus, $X_\xi \in \mathfrak{X}_L(G)$.

Therefore, $\mathfrak{X}_L(G)$ is isomorphic to $T_e G$, and as a consequence the dimension of a lie-algebra is same as the dimension of the underlying lie-group.

- Then we can define a Lie-bracket in T_eG as —

$$[\xi_1, \xi_2] = [X_{\xi_1}, X_{\xi_2}](e) \quad , \xi_1, \xi_2 \in T_eG.$$

With this construction, T_eG (the tangent space at $e \in G$) becomes a Lie-algebra. We denote it as $\mathfrak{g}$.

- If G is a matrix Lie-group (e.g. $GL(n)$, $SO(n)$, $SE(n)$), $\mathfrak{g}$ is a matrix Lie-algebra (e.g. $\mathfrak{gl}(n)$, $\mathfrak{so}(n)$, $\mathfrak{se}(n)$), and for $\xi_1, \xi_2 \in \mathfrak{g}$, we have —

$$[\xi_1, \xi_2] = \xi_1\xi_2 - \xi_2\xi_1 .$$

$$\boxed{\begin{matrix} X_\xi(g) = g\xi \\ [g\xi_1, g\xi_2]_{\mathfrak{X}_L(G)} \end{matrix}} \begin{matrix} \longleftrightarrow \\ \longleftrightarrow \end{matrix} \boxed{\begin{matrix} \xi \in \mathfrak{g} \\ [\xi_1, \xi_2]_{\mathfrak{g}} \end{matrix}}$$

- As $\dot{g} = g\xi$ can alternatively be expressed as $g^{-1}\dot{g} = \xi \in \mathfrak{g}$, we can associate a smooth curve in $\mathfrak{g}$ to each smooth curve in G (integral to some left invariant vector field). The converse can also be shown, but that involves representing the solution of the ODE $\dot{g}(t) = g(t)\,\xi(t)$ in a certain form. (See the Wei-Norman 1964 AMS Paper [Paper 15])

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Example: $g(t) = \begin{bmatrix} \cos t^2 & -\sin t^2 \\ \sin t^2 & \cos t^2 \end{bmatrix}$ is a smooth curve on $SO(2)$.

$$\dot{g}(t) = 2t\begin{bmatrix} -\sin t^2 & -\cos t^2 \\ \cos t^2 & -\sin t^2 \end{bmatrix} = 2t\begin{bmatrix} \cos t^2 & -\sin t^2 \\ \sin t^2 & \cos t^2 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$= g(t)\underbrace{\begin{bmatrix} 0 & -2t \\ 2t & 0 \end{bmatrix}}_{\xi(t) \in so(2)}$$

Example:

$\xi(t) = \begin{bmatrix} \Omega & b \\ 0 & 0 \end{bmatrix}$ $\quad \Omega = -\Omega^T \in so(n)$, $b \in \mathbb{R}^n$

$$\dot{g}(t) = g(t)\xi(t) = g(t)\begin{bmatrix} \Omega & b \\ 0 & 0 \end{bmatrix}$$ ← a dynamics on $SE(n)$

$\Downarrow$

$$g(t) = g(0)\exp\left(\begin{bmatrix} \Omega t & bt \\ 0 & 0 \end{bmatrix}\right)$$

$$= g(0)\underbrace{\begin{bmatrix} e^{\Omega t} & bt + \frac{1}{2!}\Omega b t^2 + \frac{1}{3!}\Omega^2 b t^3 + \cdots \\ 0 & 1 \end{bmatrix}}_{\in SE(n)}$$

$\xi(t) = \begin{bmatrix} \Omega & b \\ 0 & 0 \end{bmatrix} \in se(n)$ corresponds to curves on $SE(n)$

Integral Curves and Exponential Map:—

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→ Let $X \in \mathfrak{X}(M)$. An integral curve to X is a pair (I, ϕ) where $I \subset \mathbb{R}$ is an open interval and $\phi : I \to M$ is a smooth map such that —

$$\phi_{*t_0} \frac{\partial}{\partial t}\bigg|_{t_0} = X\Big|_{\phi(t_0)} \quad \text{for all } t_0 \in I.$$

→ If (I, ϕ) is an integral curve of $X \in \mathfrak{X}(M)$, then so is $(I + a, \phi_a)$ where $\phi_a(t) \triangleq \phi(t-a)$. Also, $(s^{-1}I, {}_s\phi)$ is an integral curve for sX, $s \neq 0$, where ${}_s\phi(t) \triangleq \phi(st)$. Finally, if (U, f) a local chart with coordinate functions $(x_1, \dots, x_n)$ and X can be expressed in local coordinates as —

$$X = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i}, \quad a_i \in C^\infty(U),$$

then (I, ϕ) with $\phi : I \to U \subset M$, $t \longmapsto \phi(t) = (\phi_1(t), \dots, \phi_n(t))$ is an integral curve iff —

$$\frac{d}{dt}(\phi_i(t)) = a_i(\phi(t))$$

→ Moreover, for a given (I, ϕ) there is a unique maximal extension $(\hat{I}, \hat{\phi})$ such that whenever there is an integral curve $(\hat{I}, \hat{\phi})$ s.t.

$I \subset \hat{I}$ and $\hat{\phi}|_I = \phi$ we have, $\hat{I} \subset \tilde{I}$ and $\tilde{\phi}|_{\hat{I}} = \hat{\phi}$. This is called a maximal integral curve. A vector field $X \in \mathfrak{X}(M)$ is called complete if the domains of definition ~~of~~ all of its maximal integral curves are $\mathbb{R}$.

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→ Suppose $X \in \mathfrak{X}(G)$ is a left invariant vector field on the Lie-group G. Then X is complete.

→ Then for a complete vector field $X \in \mathfrak{X}(G)$, we can define a smooth map —

$$\Phi_X : \mathbb{R} \times G \to G$$

such that — i) $\Phi_X(0, g) = g \quad \forall g \in G$

ii) $t \mapsto \Phi_X(t, g)$ is an integral curve of X for all $g \in G$.

As discussed earlier each left invariant vector field on G can be identified with the Lie-algebra $\mathfrak{g}$. Hence we can define a map —

$$\tilde{\Phi} : \mathbb{R} \times G \times \mathfrak{g} \to G$$

$$(t, g, \xi) \longmapsto \Phi_X(t, g) \quad X(g) = \tilde{g}\xi.$$

Then, clearly $\tilde{\Phi}(0, g, \xi) = g$ and $t \mapsto \tilde{\Phi}(t, g, \xi)$ is an integral curve. Moreover, $\tilde{\Phi}(t, g, \xi) = g\tilde{\Phi}(t, e, \xi)$, $\tilde{\Phi}(st, g, \xi) = \tilde{\Phi}(t, g, s\xi)$. Hence, $\boxed{\tilde{\Phi}(t, g, \xi) = g\tilde{\Phi}(1, e, t\xi)}$