Lecture 12.

MAE 544
NONLINEAR CONTROL

10/25/2017
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Let, $X \in \mathfrak{X}(M)$ be a smooth ~~vec~~ and complete vector field on the manifold M. Then we can define a smooth map —

$$\boxed{\Phi_t^X : M \longrightarrow M}$$
$$g \longmapsto \Phi_t^X(g) = \Phi_X(t, g)$$

Φ_t^X is called the flow of X at time $t \in \mathbb{R}$. On the other hand the map $t \longmapsto \Phi_t^X(g)$ defines the integral curve of X through $g \in M$.

As discussed earlier, any left invariant vector field on a Lie-group (G) can be identified with an element of the associated Lie-algebra ($\mathfrak{g} = T_eG$). This allows us to define a map —

$$\boxed{\exp : \mathfrak{g} \longrightarrow G}$$
$$\xi \longmapsto \exp(\xi) = \Phi_1^{X_\xi}(e),$$

where, the left-invariant vector-field $X_\xi \in \mathfrak{X}_L(G)$ is defined as $X_\xi(g) = g\xi$, $g \in G$.

As, $\Phi_1^{tX_\xi} = \Phi_t^{X_\xi}$ for any $t \in \mathbb{R}$, $\exp(t\xi) = \Phi_t^{X_\xi}(e)$ provides a way to define the flow corresponding to $X_\xi \in \mathfrak{X}_L(G)$.

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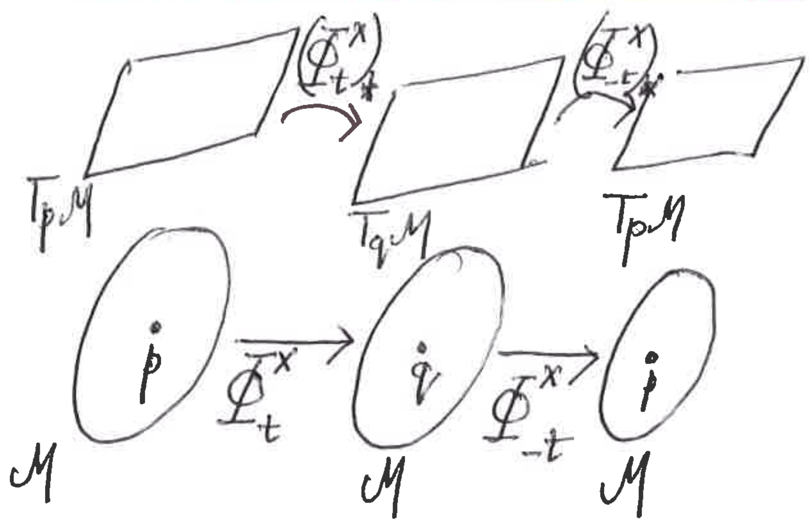
Moreover, it can be shown that — $\exp : \mathfrak{g} \to G$ is a diffeomorphism between $\mathfrak{g}$ and G. This can be leveraged to define a chart around $e \in G$ (see Lie-Cartan coordinates of the first and second kind).

Connection between Flow and Lie-Brackets: —



$X, Y \in \mathfrak{X}(M)$, $p \in M$.

Let, $\Phi_t^X(p) = q$

Then, $Y(\Phi_t^X(p)) \in T_qM$

We can show that —

$$[X,Y] = \lim_{t \to 0} \left((\Phi_{-t}^X)_* \, Y(\Phi_t^X(p)) - Y(p) \right)$$

This relationship allows us to perceive $[X,Y]$ as the derivative of Y along the direction generated by X. We also call $[X,Y]$ the Lie-derivative Y along X $(\mathcal{L}_X Y)$.

→ Another Result:

$[X,Y] = 0$ if and only if $\Phi_{t_1}^X \circ \Phi_{t_2}^Y = \Phi_{t_2}^Y \circ \Phi_{t_1}^X$ for all t_1, t_2 for which these flows are defined.

Distribution and Frobenius Theorem:-

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→ A distribution $\mathcal{D}$ on a manifold M is a map $M \ni p \longmapsto \mathcal{D}(p) \subset T_pM$, i.e. it assigns a subspace of the tangent space to each point on the manifold. More formally, for a given set of smooth vector fields $X_1, \ldots, X_m$ on a smooth manifold M, we define the associated distribution $\mathcal{D}$ as —

$$\mathcal{D} = \left\{ \phi_1 X_1 + \phi_2 X_2 + \cdots + \phi_m X_m \,\middle|\, \begin{matrix} \phi_i \in C^\infty(M) \\ 1 \leq i \leq m \end{matrix} \right\}.$$

As a consequence $\mathcal{D}$ can be viewed as a subset of $\mathfrak{X}(M)$. Also, we have —

$$\mathcal{D}(p) = \operatorname{span}\{X_1(p), X_2(p), \ldots, X_m(p)\} \subset T_pM.$$

The dimension of $\mathcal{D}$ at $p \in M$ is defined as the dimension of the subspace $\mathcal{D}(p)$.

→ A distribution $\mathcal{D}$ is called smooth if around any point $p \in M$, there exists a neighborhood U_p and smooth vector fields $\{X_\alpha^p\}_{\alpha \in A}$ such that —

$$\mathcal{D}(q) = \operatorname{span}\{X_\alpha^p(q)\}_{\alpha \in A} \quad \text{for all } q \in U_p.$$

And $\mathcal{D}$ is non-singular if the dimension of $\mathcal{D}(p)$ remains constant at every point $p \in M$. Then, $\dim(\mathcal{D}) = \dim(\mathcal{D}(p))$, $p \in M$.

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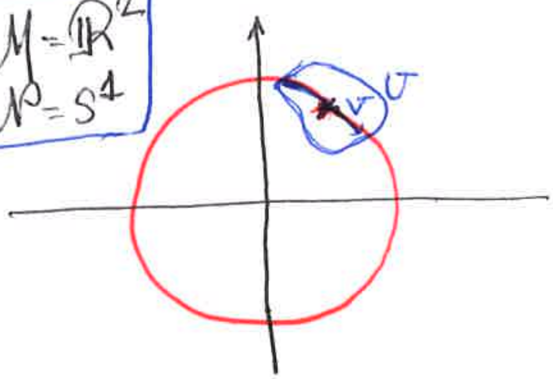
→ For a smooth distribution $\mathcal{D}$, we can also define the set —

$$\Gamma(\mathcal{D}) = \{X \in \mathfrak{X}(M) \mid X(p) = \mathcal{D}(p) \ \forall p \in M\}.$$

A distribution $\mathcal{D}$ is called to be involutive if $\Gamma(\mathcal{D})$ is closed under the Lie-bracket operation, i.e. $[X, Y] \in \Gamma(\mathcal{D})$ whenever $X, Y \in \Gamma(\mathcal{D})$.

→ Let, M be an n-dimensional manifold, and $N \subset M$ is a k-dimensional manifold. Moreover, for every $p \in N$, there is chart (U, f) on M such that $p \in U$ and an open neighborhood V of p in N, such that $U \cap V = \{q \in V \mid x_i(q) = x_i(p),\ i = k+1, \ldots, n\}$ where $(x_1, \ldots, x_n)$ are coordinate functions of (U, f).

$M = \mathbb{R}^2$
$N = S^1$



$X = \mathbb{R}$
$\Psi : t \longmapsto (\cos t, \sin t)$

Another alternative way to think about immersed submanifold: Let $X \subset M$ be a manifold of dimension k, and $\Psi : X \to M$ is smooth map. Moreover, $N = \Psi(X)$. Ψ_{*p} is one-to-one at every $p \in X$ and

If either of these conditions hold true for $N \subseteq M$, it is called an immersed submanifold of M.

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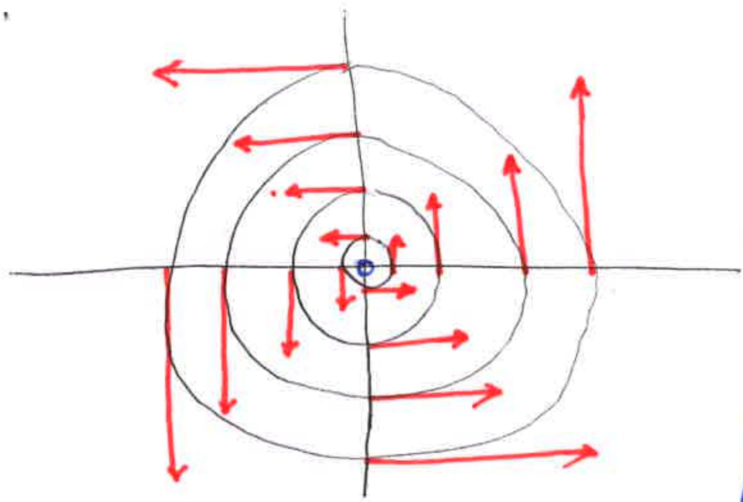
→ A distribution $\mathcal{D}$ is integrable if for all $p \in M$, there exists an immersed submanifold $N \subset M$ such that $p \in N$ and

$$T_q N = \mathcal{D}(q) \quad \text{for all } q \in N.$$

Then N is called an integral manifold of $\mathcal{D}$ (or, a leaf of $\mathcal{D}$). A collection of all such leaves of $\mathcal{D}$ forms a foliation of $\mathcal{D}$.



$$M = \mathbb{R}^2 \setminus \{0\}$$

$$\mathcal{D}(x) = \text{span}\left\{\begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}\right\}$$

$$T_x S^1_{\|x\|} = \mathcal{D}(x)$$

$$\uparrow \quad \{x \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = \|x\|^2\}$$

→ leaves of $\mathcal{D}$ are concentric circles around the origin.

Consider $M = \mathbb{R}^3 \setminus \{0\}$.

$$\mathcal{D}(x) = \text{span}\left\{\begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix}, \begin{pmatrix} -x_3 \\ 0 \\ x_1 \end{pmatrix}, \begin{pmatrix} 0 \\ -x_3 \\ x_2 \end{pmatrix}\right\}$$

Then $T_x S^2_{\|x\|} = \mathcal{D}(x)$, and leaves of $\mathcal{D}$ are concentric spheres around origin.

→ **Frobenius Theorem:**– Let $\mathcal{D}$ be a nonsingular smooth distribution on M. Then $\mathcal{D}$ is integrable if and only if $\mathcal{D}$ is involutive. Integral manifolds are unique.

Controllability of input-affine systems:—

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→ Control System:—

$$\boxed{\frac{d}{dt}\left(x(t)\right) = f\left(x(t)\right) + \sum_{i=1}^{m} u_i(t)\, g_i\left(x(t)\right)} \quad — (1)$$

— $x \triangleq (x_1, —, x_n)$: local coordinates for the state space manifold M.

— $f, g_1, g_2, —, g_m \in \mathfrak{X}(M)$

— $u_i \in \mathcal{U}$: class of admissible controls.

— $u \triangleq (u_1(t), —, u_m(t)) \in U \subset \mathbb{R}^m$

— $\mathcal{F} = \left\{ f + \sum_{i=1}^{m} u_i g_i \,\middle|\, (u_1, —, u_m) \in U \right\}$

— Assumptions:

(i) U is such that $f, g_1, —, g_m \in \mathcal{F}$.

(ii) $\mathcal{U}$ consists of piecewise constant functions. (By continuity of solutions of ODE, it can be shown that it is sufficient to consider piecewise constant inputs).

— Then solution of (1) at time $t \geq 0$ for input $u(\cdot)$ and initial condition is —

$x(t) = x(t, 0, x_0, u)$ and the reachable set from x_0 is given by —

$$R(x_0) = \left\{ \Phi^{X_k}_{t_k} \circ \Phi^{X_{k-1}}_{t_{k-1}} \circ \cdots \circ \Phi^{X_1}_{t_1}(x_0) \,\middle|\, k \geq 0 \,;\, X_1, —, X_k \in \mathcal{F} \,;\, t_1, —, t_k \geq 0 \right\}$$

$\rightarrow f(x) = \begin{pmatrix} x_2 \\ -x_1 \\ 0 \end{pmatrix} \qquad g_1(x) = \begin{pmatrix} 0 \\ x_3 \\ -x_2 \end{pmatrix} \qquad M = \mathbb{R}^3$

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As, $x^T f(x) = 0 = x^T g_1(x)$ for any $x \in M$,

$\frac{d}{dt}\left(x^T(t)\, x(t)\right) \equiv 0$. Hence,

$$\mathcal{R}(x_0) \subseteq \left\{ \xi \in \mathbb{R}^3 \,\middle|\, \xi^T \xi = x_0^T x_0 \right\}$$

trajectories can not leave the sphere.

$\rightarrow$ Let $V \subset M$ be an open set and $p, q \in V$.

- Then q is reachable from p at time T relative to V if there exists a piecewise constant control input $u(\cdot)$ such that $q = x(T, 0, p, u)$ and $x(\tau, 0, p, u) \in V$ for any $\tau \in [0, T]$.

- $\mathcal{R}^V(p, T) = \bigcup_{u(\cdot) \in \mathcal{U}} x(T, 0, p, u) \leftarrow$ set of points that are reachable from p at time $T > 0$, relative to V.

- $\mathcal{R}_T^V(p) = \bigcup_{0 < t \leq T} \mathcal{R}^V(p, t) \leftarrow$ set of points that are reachable from p within time T relative to V

- $\mathcal{R}(p) = \bigcup_{T > 0} \mathcal{R}_T^M(p) \leftarrow$ reachable set

$\rightarrow$ (1) is controllable $\longrightarrow \mathcal{R}(p) = M$ for all $p \in M$.

(2) is accessible $\longrightarrow \operatorname{int}(\mathcal{R}(p)) \neq$ empty for all $p \in M$.