Controllability definitions:—

→ (1) is accessible from $p \in M$ if $R_T^V(p)$ contains a nonempty open set of M (i.e. interior of $R_T^V(p)$ is nonempty) for all neighborhoods V of p and all $T > 0$. (p need not be within this open set)

→ If (1) is accessible from any point $p \in M$ then the system (1) is called to be Locally accessible.

→ If Interior of $R(p)$ is nonempty for every $p \in M$, (1) is called to be accessible. Note, that this is weaker than local accessibility.

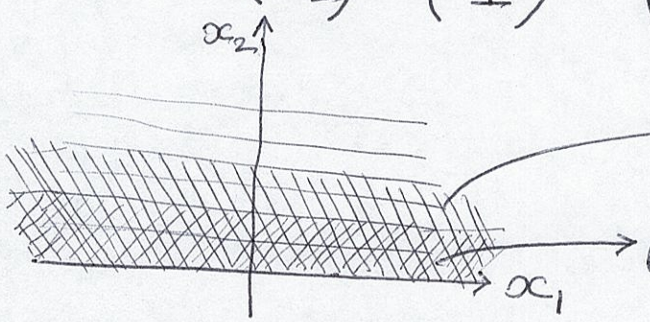
→ If $p \in \text{int.}(R(p))$, we call the system (1) to be locally controllable from $p \in M$. Moreover if $R(p) = M$, we call (1) to be globally controllable from $p \in M$. If this holds true for every $p \in M$, we call (1) to be controllable.

→ (1) is called to be small time locally controllable (STLC) from $p \in M$ if $R_T^V(p)$

contains an open neighborhood of p for all neighborhoods V of p and all $T>0$. If this holds true for every $p\in M$, we call the system to <u>small time locally controllable (STLC).</u>

→ <u>Accessible, but not Locally controllable.</u>

$$M=\mathbb{R}^2 \; ; \; \begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix}+\begin{pmatrix}1\\0\end{pmatrix}u \, , \quad U=\mathbb{R}$$



$$\mathcal{R}^M_{T_1}((0,0))\subset \mathcal{R}^M_{T_2}((0,0)) \, , \quad T_2>T_1$$

→ <u>Locally controllable, but not STLC</u>

$$M=\mathbb{R}\times S^1$$

$$\begin{pmatrix}\dot{x}\\ \dot{\theta}\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix}+\begin{pmatrix}1\\0\end{pmatrix}u \, , \quad U=[-1,1]$$

Clearly $\mathcal{R}^M_T((0,0))$ does not contain $(0,0)$ for small values of $T>0$. However for larger values of T $(0,0)\in \operatorname{int}\left(\mathcal{R}^M_T(0,0)\right)$

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→ If (1) is defined over a connected manifold $\mathcal{M}$. Then (1) is controllable, if it is STLC.

Proof: Suppose (1) is STLC, but not controllable. Then, $\mathcal{R}(p) \subset \mathcal{M}$, but $\mathcal{R}(p) \neq \mathcal{M}$ for some $p \in \mathcal{M}$. Let, z be a point on the boundary of $\mathcal{R}(p)$. As (1) is STLC, $\mathcal{R}(z)$ contains a neighborhood of $z \in \mathcal{M}$. Thus z cannot be a point on the boundary of $\mathcal{R}(p)$. This leads to a contradiction, and hence, $\mathcal{R}(p) = \mathcal{M}$. ∎

Drift Free Systems:—

$$\frac{d}{dt}\left(x(t)\right) = \sum_{i=1}^{m} u_i(t)\, g_i\left(x(t)\right)$$

— $u_i(\cdot)$ are piecewise constant functions and $u(t) \triangleq \left(u_1(t), —, u_m(t)\right) \in U = \mathbb{R}^m$ for any t.

— $\mathcal{D} = \left\{\alpha_1 g_1 + — + \alpha_m g_m \,\middle|\, (\alpha_1, —, \alpha_m) \in \mathbb{R}^m\right\} \subseteq \mathfrak{X}(\mathcal{M})$

$$\mathcal{D}(x) = \operatorname{span}\left\{g_1(x), —, g_m(x)\right\} \subset T_x\mathcal{M}$$

and, $\Delta_{\mathcal{D}} \triangleq \bigcup_{x \in \mathcal{M}} \mathcal{D}(x)$

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If $\mathcal{D}$ is integrable then $\dot{x}(t) = \mathcal{D}(x(t))$ always remains tangent to an integral manifold $N \subset M$. Hence, controllability of a system depends on the dimension of N. As integrability of a distribution is equivalent to its involutivity, controllability is related to involutivity of a distribution.

→ $M = \mathbb{R}^3 \setminus \{0\}$. $x \in M$

$$g_1(x) = \begin{pmatrix} x_2 \\ -x_1 \\ 0 \end{pmatrix} \quad g_2(x) = \begin{pmatrix} 0 \\ x_3 \\ -x_2 \end{pmatrix} \quad g_3(x) = \begin{pmatrix} x_3 \\ 0 \\ -x_1 \end{pmatrix}$$

— span $\{g_1(x), g_2(x), g_3(x)\}$ has dimension 2 at every $x \in M$.

— $[g_1, g_2] = -g_3$, $[g_2, g_3] = -g_1$, $[g_3, g_1] = -g_2$, and hence $\mathcal{D}$ is involutive.

— As $\dot{x}^T(t)\,x(t) \equiv 0$, the trajectories are confined to a sphere, making the system not accessible. However, we are yet to characterize the reachable set.

→ $M = \mathbb{R}^4 \setminus \{0\}$, $g_1(x) = \begin{pmatrix} x_4 \\ x_3 \\ -x_2 \\ -x_1 \end{pmatrix}$; $g_2(x) = \begin{pmatrix} x_3 \\ -x_4 \\ -x_1 \\ x_2 \end{pmatrix}$; $[g_1, g_2](x) = 2\begin{pmatrix} x_2 \\ -x_1 \\ x_4 \\ -x_3 \end{pmatrix}$

— dimension of span$\{g_1(x), g_2(x)\}$ is 2 at any $x \in \mathcal{M}$.

— $\mathcal{D}$ is not involutive, and hence it is not integrable. As a consequence, $\mathcal{R}(x)$ is not restricted to lie on a two-dimensional submanifold of $\mathcal{M}$.

— However, as $x^T g_1(x) = x^T g_2(x) = 0$ at any $x \in \mathcal{M}$, the system is not accessible.

(iii) Orbits of Drift-Free Systems :—

The orbit of a point $p \in \mathcal{M}$, under the distribut$\,$ $\mathcal{D}$ is —

$$\mathcal{O}(p) = \left\{ \Phi_{t_k}^{X_k} \circ \Phi_{t_{k-1}}^{X_{k-1}} \circ \cdots \circ \Phi_{t_1}^{X_1}(p) \;\middle|\; k \geq 0;\ X_1, \ldots, X_k \in \mathcal{D};\ t_1, \ldots, t_k \in \mathbb{R} \right\}$$

Clearly, $\mathcal{R}(p) \subseteq \mathcal{O}(p)$

For driftless systems with $U = \mathbb{R}^m$, we can show that $-X \in \mathcal{D}$ whenever $X \in \mathcal{D}$. Then, it readily follows that —

$$\boxed{\mathcal{R}(p) = \mathcal{O}(p)}$$

← Reachable set of $p \in \mathcal{M}$ is the orbit of p.

— $$\lim_{n \to \infty} \left[\left(\Phi_{t/n}^{X} \circ \Phi_{t/n}^{Y} \right)^n (p) \right] = \Phi_t^{X+Y}(p)$$

— $$\lim_{n \to \infty} \left[\left(\Phi_{\sqrt{t/n}}^{-Y} \circ \Phi_{\sqrt{t/n}}^{-X} \circ \Phi_{\sqrt{t/n}}^{Y} \circ \Phi_{\sqrt{t/n}}^{X} \right)^n (p) \right] = \Phi_t^{[X,Y]}(p)$$

→ Let, $\mathcal{D} = \{X, Y\} \subset \mathfrak{X}(M)$.

— Then $\Phi_t^{X+Y}(p)$ and $\Phi_t^{[X,Y]}(p)$ belong to the closure of the ~~orbits~~ orbit of $p \in M$ under $\mathcal{D}$.

→ This provides us motivation to consider the Lie-algebra generated by $\mathcal{D}$, i.e. the expansion of $\mathcal{D}$ by recursively including the sums and brackets of vector fields of $\mathcal{D}$ into $\mathcal{D}$.

→ Accessibility Lie-algebra : $\mathcal{L}(\mathcal{D})$

$\mathcal{L}(\mathcal{D})$ = The smallest Lie-subalgebra of $\mathfrak{X}(M)$ which contains $\mathcal{D}$.

- $\mathcal{L}(\mathcal{D})$ = real linear span of expressions of the form —

$$[X_k, [X_{k-1}, \cdots, [X_2, X_1]] \cdots]$$

where, $1 \leq k < \infty$

$$X_i \in \{g_1, \cdots, g_m\} \quad 1 \leq i \leq k.$$

Clearly, $(\mathcal{L}(\mathcal{D}))(p)$ contains all the tangent vectors in $\mathcal{D}(p)$. Also, by definition, $\mathcal{L}(\mathcal{D})$ is involutive, and hence trajectories passing through $p \in M$ are contained in the integral manifold of $\mathcal{L}(\mathcal{D})$ through p, if dimension of

$(\mathcal{L}(\mathcal{D}))(q)$ is same for every $q \in M$.

$\mathcal{R}(p) \subseteq$ orbit of p under $\mathcal{D}$
$\subseteq$ orbit of p under $\mathcal{L}(\mathcal{D})$
$\subseteq$ integral manifold of $\mathcal{L}(\mathcal{D})$ through p.

→ Chow's Theorem: Suppose $\mathcal{L}(\mathcal{D})$ has the same dimension at every point on M. Then the orbit of p under $\mathcal{D}$ is the integral manifold of $\mathcal{L}(\mathcal{D})$ through p.

→ Suppose $\dim\big(\mathcal{L}(\mathcal{D})(p)\big) = k$ at every $p \in M$. Then the system is controllable iff $k = n = \dim(M)$.