Lecture 14

# MAE 544
# NONLINEAR CONTROL

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## Controllability of Driftless Systems:—

$$\boxed{\dot{x}(t) = \sum_{i=1}^{m} u_i(t)\, g_i\big(x(t)\big)} \qquad (1)$$

— $(u_1(t), \ldots, u_m(t)) \in \mathbb{R}^m$ at any $t$ and $x(t) \in M$.

— $\mathcal{D} = \text{span}\{g_1, g_2, \ldots, g_m\}$ and $\mathcal{L}(\mathcal{D})$ is the smallest lie-subalgebra which contains $\mathcal{D}$.

— The system (1) is controllable if and only if $\dim\big(\mathcal{L}(\mathcal{D})(p)\big) = \dim(M)$ at every $p \in M$, whenever $M$ is connected.

## Controllability for Systems w/ Drift:—

$$\boxed{\dot{x} = f(x) + \sum_{i=1}^{m} u_i\, g_i(x)} \qquad (2)$$

— $x \in M$ and $u = (u_1, \ldots, u_m) \in U \subseteq \mathbb{R}^m$ at any $t$.

— $\mathcal{F} = \{ f + \sum_{i=1}^{m} u_i g_i \mid (u_1, \ldots, u_m) \in U \} \subseteq \mathfrak{X}(M)$

→ **Local result on controllability: (STLC)**

Let, $x_0 \in M$ be such that $f(x_0) = 0$ and $U$ contain an open neighborhood $V$ such that $(u_1, \ldots, u_m) = 0 \in V \subset U$.

If the linearization of (2) around $(x_0, 0)$ is controllable, i.e. if the pair $\big(f'(x_0), [g_1(x_0) \ldots g_m(x_0)]\big)$ is controllable, then for every $T > 0$ and $\varepsilon > 0$ the set of points reachable from $x_0$ in time $T$ using admissible controls $u(\cdot) : [0, T] \to V$ with $\|u(t)\| \leq \varepsilon$ for all $t \in [0, T]$ contains a neighborhood of $x_0$.

## Accessibility Lie-algebra —

The accessibility (control) Lie-algebra for (2) is defined as the smallest Lie-subalgebra of $\mathfrak{X}(M)$ which contains $\{f, g_1, \ldots, g_m\}$ and we denote it as $\mathcal{L}(\mathcal{F})$. Similar to the driftless scenario, $\mathcal{L}(\mathcal{F})$ is the real linear span of brackets of the form —

$$[X_k, [X_{k-1}, \ldots [X_2, X_1] \ldots]]$$

where, $1 \leq k < \infty$, $X_i \in \{f, g_1, \ldots, g_m\}$ for $i = 1, \ldots, k$; and $k = 1$ is the no bracket case.

Then, we can define the accessibility distribution as —

$$\mathcal{C}(x) = \text{span}\{X(x) \mid X \in \mathcal{L}(\mathcal{F})\} \quad x \in M.$$

$\mathcal{C}$ is a smooth distribution

## Accessibility Theorem :—

If $\dim \mathcal{C}(x_0) = n = \dim(M)$, then (2) is locally accessible from $x_0 \in M$, i.e. $\mathcal{R}^V_T(x_0)$ contains a non-empty open set of $M$ for all $T > 0$ and for all neighborhoods $V$ of $x_0$.

Moreover, if $\dim \mathcal{C}(x) = \dim(M)$ for all $x \in M$, (2) is locally accessible. This condition is called the Lie-algebraic rank condition (LARC).

This result can be proved by considering the flows of vector fields in $\mathcal{L}$ to create seq. of submanifolds of increasing dimension.

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Conversely, if (2) is locally accessible then $\dim C(x) = \dim(M)$ for all $x$ in an open and dense subset of $M$.

→ A stronger form of accessibility :-

A system is locally strongly accessible from $x_0 \in M$ if for any neighborhood $V$ of $x_0$, the set $\mathcal{R}^V(x_0, T)$ contains a non-empty ~~[illegible]~~ set for any sufficiently small $T > 0$. If this holds true for every $x_0 \in M$, then the system is called Locally strongly accessible.

Consider the system : $\boxed{\begin{aligned} \dot{x}_1 &= 1 \\ \dot{x}_2 &= u \end{aligned}}$ $\Big\}$ Locally accessible ✓ / Locally strongly accessible ✗

By letting $x_0 = \begin{pmatrix} x_0^1 \\ x_0^2 \end{pmatrix}$ we have —

$$\mathcal{R}^{\mathbb{R}^2}(x_0, T) = \left\{ \begin{pmatrix} x_0^1 \\ x_0^2 \end{pmatrix} + \begin{pmatrix} T \\ \int_0^T u(\tau)\,d\tau \end{pmatrix} \middle| \, u \in \mathcal{U} \right\}$$

$$\subseteq \left\{ \begin{pmatrix} x_0^1 + T \\ q \end{pmatrix} \middle| \, q \in \mathbb{R} \right\}$$

Does ~~[illegible]~~ not contain any open set

However,

$$\mathcal{R}_T^{\mathbb{R}^2}(x_0) = \left\{ \begin{pmatrix} q^1 \\ q^2 \end{pmatrix} \middle| \, x_0^1 < q^1 \le x_0^1 + T, \; q^2 = x_0^2 + \int_0^{\tau} u(\sigma)\,d\sigma, \; u \in \mathcal{U}, \; 0 < \tau \le T \right\}$$

contains open set for any $T > 0$.

→ Now we define the strong accessibility Lie-algebra for (2) as the smallest Lie subalgebra of $\mathfrak{X}(M)$ which contains $\{g_1, \ldots, g_m\}$ ~~[illegible]~~

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and is closed under Lie-brackets with $f$. By letting $\mathcal{L}_0(\mathcal{F})$ denote this strong accessibility Lie-algebra, we have —

$$g_i \in \mathcal{L}_0(\mathcal{F}) \quad \text{for every} \quad i \in \{1, \dots, m\},$$

and, $[f, X] \in \mathcal{L}_0(\mathcal{F})$ for every $X \in \mathcal{L}_0(\mathcal{F})$.

Similar to the previous cases, $\mathcal{L}_0(\mathcal{F})$ can be expressed as real linear span of the following brackets —

$$[X_k, [X_{k-1}, \dots [X_1, g_j] \dots ]] \qquad \begin{array}{l} 1 \leq j \leq m \\ 0 \leq k < \infty \\ X_i \in \{f, g_1, \dots, g_m\} \\ i \in \{1, \dots, k\} \end{array}$$

Finally we define the strong accessibility distribution as —

$$C_0(x) = \text{span}\{X(x) \mid X \in \mathcal{L}_0(\mathcal{F})\} \quad x \in M.$$

→ If $\dim C_0(x_0) = n = \dim(M)$, then (2) is ~~strongly~~ locally strongly accessible from $x_0 \in M$, i.e. $R^V(x_0, T)$ contains a non-empty open set of $M$ for any neighborhood $V$ of $x_0$ and any $T > 0$ sufficiently small. Moreover, if $\dim C_0(x) = \dim(M)$ for every $x \in M$, then (2) is called to be locally strongly accessible. The converse is also true almost everywhere, i.e. if (2) is locally strongly accessible then $\dim C_0(x) = \dim(M)$ for all $x$ in an open and dense subset of $M$.

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## More results on Controllability:

— $\mathcal{L}$ is called symmetric when for every $X \in \mathcal{L}$, $-X$ also belongs to $\mathcal{L}$.

For example, by assuming $U = \mathbb{R}^m$ and by letting $f(x) \in \text{span}\{g_1(x), \ldots, g_m(x)\}$ we can ensure that $\mathcal{L}$ is symmetric.

— Now suppose $f(x) \in \text{span}\{g_1(x), \ldots, g_m(x)\}$ and $\mathcal{L}$ is symmetric for (2). Then, the following holds true —

(i) If $\dim \mathcal{C}(x_0) = \dim(M)$, then (2) is STLC at $x_0 \in M$.

(ii) If $\dim \mathcal{C}(x) = \dim(M)$ at every $x \in M$, then (2) is controllable whenever $M$ is connected.

⊛ For special case when $f = 0$, this results gives back the controllability results for driftless systems.

## An example:—

$$M = \mathbb{R}^3 \qquad m = 2$$

$$\begin{aligned} \dot{x} &= yz \\ \dot{y} &= u_1 - xz \\ \dot{z} &= u_2 - u_1 \end{aligned} \iff \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \underbrace{\begin{pmatrix} yz \\ -xz \\ 0 \end{pmatrix}}_{f} + \underbrace{\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}}_{g_1} u_1 + \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{g_2} u_2$$

Then,

$$[f, g_1] = \begin{bmatrix} y - z \\ -x \\ 0 \end{bmatrix} \qquad [f, g_2] = \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix} \qquad [g_1, g_2] = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$[g_1, [f, g_1]] = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \; \text{--- and so on.}$$

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— $C_0(x) = \mathbb{R}^3$ at every $x \in M = \mathbb{R}^3$

— This system is locally strongly accessible.

— $C(x) = \mathbb{R}^3$ at every $x \in M = \mathbb{R}^3$

— This system is locally accessible.

Linearized system at $((x,y,z),(0,0))$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} 0 & z & y \\ -z & 0 & -x \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} u_1 + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u_2$$

$$A = \begin{bmatrix} 0 & z & y \\ -z & 0 & -x \\ 0 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ -1 & 1 \end{bmatrix}$$

— Check controllability of $(A,B)$ pair at a point $(x,y,z) \in \mathbb{R}^3$.

→ Another example: Controllability of underactuated Rigid body dynamics.

$$\dot{\omega}_1 = \left(\frac{I_2 - I_3}{I_1}\right)\omega_2\omega_3 + u_1$$

$$\dot{\omega}_2 = \left(\frac{I_3 - I_1}{I_2}\right)\omega_3\omega_1 + u_2$$

$$\dot{\omega}_3 = \left(\frac{I_1 - I_2}{I_3}\right)\omega_1\omega_2 \longleftarrow \text{no-torque}$$

$$f = \begin{pmatrix} a\,\omega_2\omega_3 \\ b\,\omega_3\omega_1 \\ c\,\omega_1\omega_2 \end{pmatrix} \qquad g_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad g_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

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$$[f, g_1] = -\frac{\partial f}{\partial \omega} g_1 = \begin{bmatrix} 0 \\ -b\omega_3 \\ -c\omega_2 \end{bmatrix}$$

$$[f, g_2] = -\frac{\partial f}{\partial \omega} g_2 = \begin{bmatrix} -a\omega_3 \\ 0 \\ -c\omega_1 \end{bmatrix}$$

$$[g_2, [f, g_1]] = \begin{bmatrix} 0 \\ 0 \\ c \end{bmatrix}$$

$$[g_1, [f, g_1]] = 0 = [g_2, [f, g_2]]$$

$$[g_1, [f, g_2]] = \begin{bmatrix} 0 \\ 0 \\ c \end{bmatrix}$$

Thus this dynamics is locally strongly accessible if $c \neq 0$, i.e. if $I_1 \neq I_2$.

$\longrightarrow$ Symmetry in the dynamics is undesirable from a controllability perspective.