MAE 544

NONLINEAR CONTROL

- Here we consider a dynamic system of the form —

$$\boxed{\dot{x} = f(x, t)} \text{ with } x \in \mathbb{R}^n, \quad x(t_0) = x_0 \quad (1)$$

We also assume that the function $f: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is Lipschitz continuous with respect to x uniformly in t (i.e. the Lipschitz constant does not depend on t), and piecewise continuous in t. This ensures that solutions exist for the ODE given in (1) and they are unique.

→ $x^* \in \mathbb{R}^n$ is an equilibrium of (1) if $f(x^*, t) \equiv 0$ for any time t.

Now, let's define, $\xi \triangleq x - x^*$

then, $\xi(t_0) = x_0 - x^*$

and, $\dot{\xi} = f(\xi + x^*, t) \triangleq g(\xi, t)$.

Clearly $g(0, t) = f(x^*, t) \equiv 0$

Therefore, $\xi^* = 0$ is an equilibrium for the dynamic system —

$$\boxed{\dot{\xi} = g(\xi, t)} \quad \xi(t_0) = x_0 - x^*.$$

This allows us to assume $x^* = 0$ to be an equilibrium of (1) without any loss of generality

Stability in the sense of Lyapunov:-

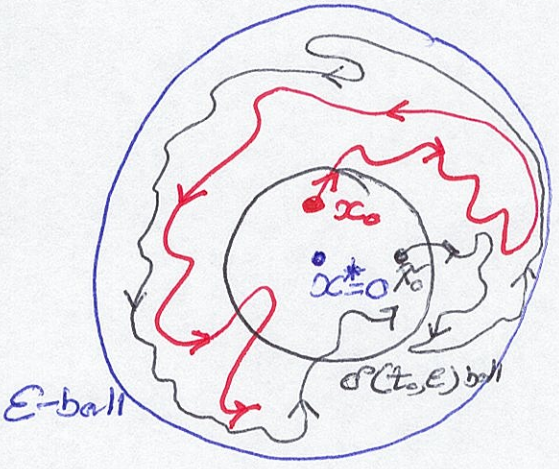
This informally means that trajectories that start close to an equilibrium always remain close to that equilibrium. Although this perspective towards stability does not take an external input into consideration, it provides ~~& a tool~~ tools to analyze the closed-loop behavior of the system. In what follow we will consider the dynamics —

$$\boxed{\dot{x} = f(x,t)} \qquad x(t_0) = x_0 \qquad \text{———} \quad (A)$$

where $x^* = 0$ is an equilibrium of (A) and f satisfies the properties which ensure existence and uniqueness of solutions for (A).

→ The equilibrium point $x^* = 0$ of (A) is "stable" if for all $t_0 > 0$ and $\varepsilon > 0$, there exists a $\delta(t_0, \varepsilon) > 0$ such that —

$$\|x_0\| < \delta(t_0,\varepsilon) \Rightarrow \|x(t) < \varepsilon \quad \forall t \geq t_0.$$



Clearly, ε should be atleast as large as $\delta(t_0, \varepsilon)$. Also, this notion does not prevent $\delta(t_0, \varepsilon)$ from shrinking when t_0 changes for a fixed $\varepsilon > 0$.

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Consider a scalar system whose dynamics is given by —

$$\boxed{\dot{x} = (6t\sin t - 2t)x}, \quad x(t_0) = x_0$$

Then we can show that corresponding state trajectory can be expressed as —

$$x(t) = x_0 \exp\left(6(\sin t - t\cos t) - t^2 - 6(\sin t_0 - t_0\cos t_0) + t_0^2\right)$$

$$\Rightarrow \|x(t)\| \leq \|x_0\| \underbrace{\left\|\exp\left(-6\sin t_0 + 6t_0\cos t_0 + t_0^2 + \overset{\text{a const}}{K}\right)\right\|}_{\triangleq\, c(t_0)}$$

clearly $c(t_0)$ grows as t_0 increases.

Now, for a given $\varepsilon > 0$ and $t_0 > 0$, we can choose, $\delta(t_0, \varepsilon) = \dfrac{\varepsilon}{c(t_0)}$.

Then,

$$\|x(t)\| \leq \|x_0\|\, c(t_0)$$

$$< \varepsilon \quad \text{whenever } \|x_0\| < \delta(t_0, \varepsilon) = \frac{\varepsilon}{c(t_0)}$$

As a consequence, the trajectories that start later (i.e. have high t_0) can have go further away from the equilibrium at $x^* = 0$.

→ The equilibrium point $x^* = 0$ of (A) is "uniformly stable" if $\forall\, t_0 > 0$ and $\varepsilon > 0$ there exists a $\delta(\varepsilon)$ such that —

$$\|x_0\| < \delta(\varepsilon) \Rightarrow \|x(t)\| < \varepsilon \quad \forall\, t \geq t_0.$$

Clearly, for time-invariant dynamics (i.e. when $\dot{x} = f(x)$) stability and uniform stability are the same.

Finally, $x^* = 0$ is an "unstable equilibrium point" of (A) if it is not stable.

However, neither stability nor uniform stability provides any insight about whether a solution trajectory converges to the equilibrium.

→ $x^* = 0$ is an "asymptotically stable (A.S.) equilibrium" point of (A) if —

(i) $x^* = 0$ is a stable equilibrium point.

(ii) $x^* = 0$ is attractive, i.e. for all $t_0 > 0$, there exists $\delta(t_0) > 0$ such that —

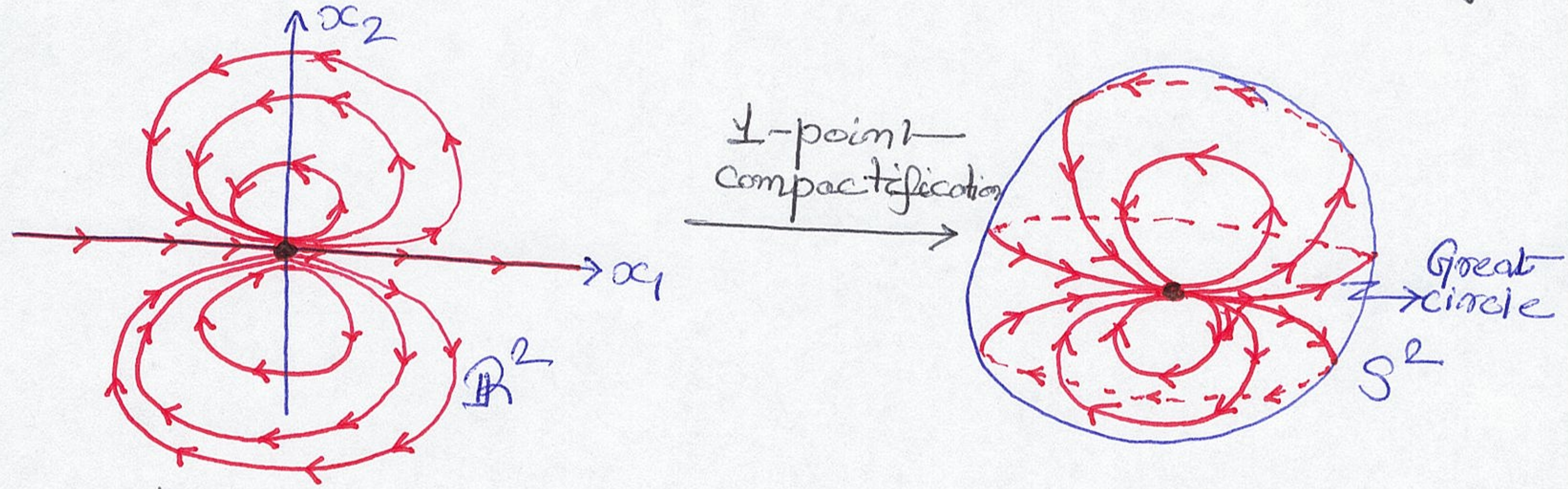
$$\|x_0\| < \delta \implies \|x(t)\| \to 0 \text{ as } t \to \infty.$$

- It should be noted that an equilibrium may not be stable even if each trajectory converges to the equilibrium. This is the reason that we need two separate conditions to define asymptotic stability. The following example highlights this aspect.

Consider the dynamics —

$$\boxed{\begin{aligned} \dot{x}_1 &= x_1^2 - x_2^2 \\ \dot{x}_2 &= 2x_1 x_2 \end{aligned}} \quad \text{————} \quad (*)$$

Then all trajectories converge to the origin, except the trajectories originating from the positive x_1-axis. However by identifying the points at infinity (this can be done by viewing the plane $\mathbb{R}^2$ as the stereographic projection of S^2 [this process is called 1-point compactification of $\mathbb{R}^2$|), we can conclude that every trajectory converges to origin. But the origin is not stable, because irrespective of our choice of $\varepsilon > 0$, some trajectories will always leave the $\delta(\varepsilon)$ ball around the origin.



→ $x^* = 0$ is an "uniformly asymptotically stable (U.A.S.) equilibrium" point of (A) if —

(i) $x^* = 0$ is an uniformly stable equilibrium point.

(ii) The trajectory $x(t)$ converges uniformly to $x^* = 0$ whenever $\|x_0\| < \delta$, $\delta > 0$, i.e. there exists a function $\gamma : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ with $\gamma(t, \xi_0) \to 0$ as $t \to \infty$ for all $\|\xi_0\| < \delta$ such that —

$$\|x_0\| < \delta \Rightarrow \|x(t)\| \leq \gamma(t - t_0, x_0) \quad \forall\, t \geq t_0.$$

→ $x^* = 0$ is an "exponentially stable equilibrium" point of (A) if there exists positive constants $m, \alpha, \delta > 0$ such that —

$$\|x_0\| < \delta \Longrightarrow \|x(t)\| \leq m e^{\alpha(t - t_0)} \|x_0\| \quad \forall\, t \geq t_0.$$

The constant $\alpha > 0$ provides an estimate of the rate of convergence.

Please note that all these notions of stability are local in nature.

→ $x^*=0$ is a "globally asymptotically stable (G.A.S.) equilibrium" point of (A) if it is asymptotically stable (A.S.) and for any $x_0 \in \mathbb{R}^n$

$$\|x(t)\| \to 0 \text{ as } t \to \infty.$$

In a similar spirit, $x^*=0$ is a "globally uniform asymptotically stable (G.U.A.S.) equilibrium" point of (A) if $x^*=0$ is globally asymptotically stable (G.A.S.) and convergences uniformly in time, i.e. there exists a function $\gamma : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ with $\gamma(t,\xi) \to 0$ for all $\xi \in \mathbb{R}^n$ as $t \to 0$, such that —

$$\|x(t)\| < \gamma(t-t_0, x_0) \;\; \forall\, t \geq t_0.$$

Stability of Solution trajectories :—

Let, $w(t)$ be a reference soln of (A). Then for any other solution trajectory $x(t)$ we can define,

$$\xi(t) = x(t) - w(t)$$

Then,

$$\begin{aligned}\dot{\xi}(t) &= f(x(t),t) - f(w(t),t) \\ &= f(\xi(t)+w(t),t) - f(w(t),t] \\ &\triangleq g(\xi(t),t)\end{aligned}$$

Clearly, $g(0,t) = f(w(t),t) - f(w(t),t) = 0$.

Therefore, $\xi^*=0$ is an equilibrium point for the dynamic system —

$$\mathring{\xi}(t) = g(\xi(t), t).$$

Thus the stability properties of an equilibrium point can be traslated to yield/define stability properties of reference solution trajectories. This is useful for investigating stability of desired motions in robotics or stability of limit cycles/periodic orbits.

Energy-like Functions :—

→ $\alpha : [0,\infty) \to [0,\infty)$ is in class $\mathcal{K}$ if —

(i) α is continuous,
(ii) $\alpha(0) = 0$, and
(iii) α is strictly increasing.

→ $\alpha : [0,\infty) \to [0,\infty)$ is in class $\mathcal{K}_\infty$ if —

(i) α is in class $\mathcal{K}$, and
(ii) $\alpha(x) \to \infty$ as $t \to \infty$ (i.e no levelling off).

→ A continuous function $V : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}_+$ is a "locally positive definite function (L.P.D.F.)" if there exists $\varepsilon > 0$ and a class $\mathcal{K}$ function α such that —

(i) $V(0,t) = 0 \quad \forall t \geq 0$
(ii) $V(x,t) \geq \alpha(\|x\|) \quad \forall \|x\| < \varepsilon$ and $t \geq 0$.

→ A continuous function $V : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}_+$ is a "positive definite function (P.D.F.)" if there is a class $\mathcal{K}_\infty$ function α such that—

(i) $V(0,t)=0 \;\; \forall\, t \geq 0$

(ii) $V(x,t) \geq \alpha(\|x\|) \;\; \forall\, x \in \mathbb{R}^n, t \geq 0$.

→ A continuous function $V: \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}_+$ is a "decresecent function" if ~~for some~~ there exists $\varepsilon > 0$ and a class $\mathcal{K}$ function β such that—

$$V(x,t) \leq \beta(\|x\|) \quad \forall\, \|x\| < \varepsilon,\; t \geq 0.$$

Direct Method of Lyapunov:—

This provides a way to investigate ~~of~~ stability properties of a dynamic system without explicitly ~~xxx~~ solving the dynamics. Towards this objective we first define a non-negative function $V(x,t)$. Time derivative of V along trajectories of the system (A) is defined as—

$$\dot{V}(x,t) = \frac{\partial V}{\partial x}(x,t)\, f(x,t) + \frac{\partial V}{\partial t}(x,t).$$

Then following statements hold true:—

(i) $V(x,t)$ is L.P.D.F. and $\dot{V}(x,t) \leq 0$ locally in x (i.e. $\forall\, x \in B_\varepsilon = \{x \in \mathbb{R}^n / \|x\| < \varepsilon\}$) and for all $t \geq 0$, then $x^* = 0$ is a "stable" equilibrium point.

(ii) $V(x,t)$ is L.P.D.F and decrescent, and $\dot{V}(x,t) \leq 0$ for all $x \in B_\varepsilon$ and $t \geq 0$. Then $x^* = 0$ is "uniformly stable.

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(iii) $V(x,t)$ is L.P.D.F and decrescent, and $-\dot{V}(x,t)$ is L.P.D.F. Then, $x^*=0$ is uniformly asymptotically stable (U.A.S.).

(iv) $V(x,t)$ is P.D.F. and decrescent, and $-\dot{V}(x,t)$ is P.D.F. Then, $x^*=0$ is globally uniformly asymptotically stable (G.U.A.S.).

(v) $V(x,t)$ is such that there exists $\varepsilon > 0$ and

— $\alpha_1 \|x\|^2 \leq V(x,t) \leq \alpha_2 \|x\|^2 \quad \forall t \geq 0$

— $\dot{V}(x,t) \leq -\alpha_3 \|x\|^2 \quad \forall t \geq 0$

— $\left\| \frac{\partial V}{\partial x}(x,t) \right\| \leq \alpha_4 \|x\| \quad \forall t \geq 0$

for some positive constants $\alpha_1, \alpha_2, \alpha_3, \alpha_4 > 0$, and $\|x\| < \varepsilon$. Then $x^*=0$ is an "exponentially stable equilibrium" point of (A).

It should be noted that although these conditions are easy to check, there is no straightforward way to find or compute the nonnegative function $V(x,t)$ (these functions are called Lyapunov function for a given system) which satisfies the appropriate properties for a given system. This makes it very difficult to use Lyapunov's direct method in practice. Although there are converse theorems, but none of them are constructive.