Lect 16.

Indirect Method Lyapunov.

→ $V(x,t)$ is hard to find; on the other had for stable linear systems we can obtain quadratic Lyapunov functions by an algorithmic way. Idea is to consider the linearized dyna around an eq. If it is stable find Lyap and then go back to nL-dyna and find the region where it works.

→ Fund Th. of integral calc

X, Y : two finite dim. and U is an open subset of X. $f: U \to Y$ is cont. differentiable. i.e is C^1. If $x+ty \in U$ $\forall t \in [0,1]$, then —

$$f(x+y) = f(x) + \int_0^1 \underbrace{Df(x+ty)}_{\text{fréchet derivative}} y\, dt$$

$$Df(z)h = \frac{d}{ds}\Big|_{s=0} f(z+sh)$$

$$\boxed{f(x) = f(0) + \int_0^1 Df(tx)\, x\, dt}$$

$$= f(0) + \left(\int_0^1 Df(tx)\, dt\right) x$$

$$= f(0) + M(x)\, x$$

Consider a nl-system —

$$\dot{x} = f(x), \quad x(t_0) = x_0 \qquad \text{——} \quad (1)$$

and assume $x^* = 0$ is an equilibrium for (1), i.e $f(0) = 0$. f is C^1 (i.e cont. diff)

Then the linearized dyna around eq —

$$\dot{z} = Az \quad \text{where } A = \left(\frac{\partial f}{\partial x}\right)\Big|_{x=0}$$

$$= Df(0)$$

Express (1) as —

$$\dot{x} = Ax + \underbrace{\left(f(x) - Ax\right)}_{g(x)}$$

$\rightarrow$ $\underline{g(x) = g(0) + N(x)x}$ where $N(x) = \int_0^1 \left(Df(tx) - A\right)dt$

Now,

$$g(0) = 0$$

$$\lim_{x \to 0} N(x) = \int_0^1 \left(\lim_{x \to 0} Df(tx) - A\right)dt = 0$$

$$\|g(x)\| \leq \cancelto{=0}{\|g(0)\|} + \|N(x)x\| \leq \|N(x)\| \, \|x\|$$

$$\frac{\|g(x)\|}{\|x\|} \leq \|N(x)\| \longrightarrow 0 \quad \text{as } \|x\| \to 0 \ \underline{\text{in any norm}}$$

Then, for any arbitrary small $\gamma > 0$, $\exists$ an $r > 0$ s.t.,

$$\|g(x)\|_2 \leq \gamma \|x\|_2 \quad \forall \, \|x\|_2 < r$$

Main Thm: Let $x^* = 0$ be an eq. pt. of (1), where f is C^1 on a neighborhood D of the origin. Let $A = \left.\frac{\partial f}{\partial x}\right|_{x=0}$. Then the origin is ~~as~~ A.S. if A is Hurwitz i.e. $\mathrm{spec}(A) \subseteq \mathbb{C}^-$ the open L.H.P.

Proof As A is Hurwitz, there exists a unique $P > 0$ s.t.

$$A^T P + PA = -Q$$

for any $Q = Q^T > 0$.

$$\underbrace{P}_{\text{Sym}} = \int_0^\infty e^{A^T\tau} Q e^{A\tau}\, \underbrace{d\tau}_{\text{conv.}}$$

$$(I_n \otimes A + A^T \otimes I_n)\,\mathrm{vec}(P) = -\mathrm{vec}(Q)$$

Define, $\boxed{V(x) = x^T P x}$

P.D.F. — $\geq \lambda_{min}(P)\|x\|^2$

Decrescent $\leq \lambda_{max}(P)\|x\|^2$

$$\dot{V} = x^T P (Ax + g(x)) + (Ax + g(x))^T P x$$

$$= x^T (PA + A^T P) x + x^T P g(x) + g(x)^T P x$$

$$= -x^T Q x + 2 x^T P g(x)$$

$$x^T P g(x) \leq \|x^T P g(x)\|$$

$$\leq \|x\|_2 \cdot \|P g(x)\|$$

$$\leq \|x\|_2 \|P\|_2 \|g(x)\|$$

If $\|x\|_2 < \gamma$ then —

$$x^T P x < \|x\|_2^2 (\gamma \|P\|_2)$$

$$0 < \lambda_{min}(Q)\|x\|_2^2 \leq x^T Q x \leq \lambda_{max}(Q)\|x\|_2^2$$

$$-x^T Q x \leq -\lambda_{min}(Q)\|x\|^2$$

$$\therefore \dot{V} < -\left(\lambda_{min}(Q) - 2\gamma\|P\|_2\right)\|x\|_2^2$$

Fix $Q \Rightarrow$ fixed P. and fixed $\lambda_{min}(Q)$.

Then we have to choose $\gamma > 0$ s.t.

$$\gamma < \frac{1}{2}\frac{\lambda_{min}(Q)}{\|P\|_2}$$

$$\boxed{\begin{array}{l} \Omega = \{x \mid \Phi(x,t) \to 0\} \quad \Phi_t^f(x) \\ \text{— connected} \\ \text{— invariant.} \end{array}}$$

Then $\gamma > 0$ gives an estimate of ~~region o~~ attraction. A larger γ is allways better.

→ A has some eigenvalues in $\mathbb{C}^+$, $x = 0$ is unstable.

→ A has some eig. on im-axis. are critical cases and nothing can be said about stability via linearization.

(4) For non-aut case —

$$\dot{x} = f(t, x)$$
$$= A(t)x + g(t, x)$$

$g(t,x) \leftarrow f(t, x) - A(t)x$

$A(t) \leftarrow \frac{\partial f}{\partial x}(x,t)\Big|_{x=0}$

In general

$$\frac{\|g(t,x)\|_2}{\|x\|_2} \to 0 \quad \text{as} \quad \|x\|_2 \to 0$$

at each $t \geq 0$

But not hold uniformly.

Uniform Order Cond.

$$\lim_{\|x\|_2 \to 0} \left(\sup_{t \geq 0} \frac{\|g(t,x)\|_2}{\|x\|_2} \right) = 0$$

$$\boxed{\dot{x} = -x + tx^2}$$

Fb stabilization.

$$\left.\begin{aligned} \dot{x} &= f(x, u) \\ u &= Kx \end{aligned}\right\rangle \dot{x} = f(x, Kx)$$

$(A + BK)$

$$g(x) = f(x, Kx) - (A + BK)x$$

Example. $\dot{x}_1 = -x_2$

$\dot{x}_2 = x_1 + (x_1^2 - 1)x_2$

$(0,0)$ — eq.

$A = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$

$Q = I \implies P = \begin{pmatrix} 1.5 & -0.5 \\ -0.5 & 1 \end{pmatrix}$

$V(x) = x^T P x = \frac{1}{2}(3x_1^2 - 2x_1x_2 + x_2^2)$

$\dot{V}(x) = -(x_1^2 + x_2^2) - (\underbrace{x_1^3 x_2 - 2x_1^2 x_2^2}_{x_1(x_1x_2)(x_1 - 2x_2)})$

$\leq \|x\|^2 + |x_1|\,|x_1x_2|\,|x_1 - 2x_2|$

$|x_1 - 2x_2| \leq \sqrt{5}\|x\|$

$|x_1x_2| \leq \frac{1}{2}\|x\|^2$

$|x_1| \leq \|x\|$

$\leq -\|x\|^2\left(1 - \frac{\sqrt{5}}{2}\|x\|^2\right)$

$1 - \frac{\sqrt{5}}{2}\|x\|^2 > 0 \longrightarrow \|x\|^2 \leq \frac{2}{\sqrt{5}} = \sqrt{.8994} = .9457$

$c < \min V(x)$

$x_1 = \rho c\theta$

$x_2 =$

$\dot{V} \leq -\rho^2 + .861\rho^4$

$\rho^2 < \frac{1}{0.861}$