(iii) Input Output stability dates back the idea of internal or Lyapunov stability. The def. and th. from state space lead to correspondin ip/op notions, However the converse is not true in general. We will need additional hypotheses.

$$\boxed{\begin{aligned} \dot{x}_1 &= -x_1 + u \\ \boxed{\dot{x}_2} &\boxed{= x_1^2 + x_2^2} \\ y &= x_1 \end{aligned}}$$

One idea/type of external stability says bounded i/p shall always give bdd. o/p.

$$\cancel{x_1(t_0)} = \frac{dx}{dt} = -x_1 + u$$

$$\Rightarrow dx + x\,dt = u\,dt$$

$$\Rightarrow e^{t}dx + e^{t}dt\,x = e^{t}u\,dt$$

$$\Rightarrow e^{t}x(t) - e^{t_0}x_0 = \int_{t_0}^{t} e^{\tau}u(\tau)\,d\tau$$

$$\Rightarrow x_1(t) = e^{-(t-t_0)}x_1(t_0) + \int_{t_0}^{t} e^{(\tau-t)}u(\tau)\,d\tau$$

$$y(t) = e^{-(t-t_0)}x_1(t_0) + \int_{t_0}^{t} e^{-(t-\tau)}u(\tau)\,d\tau$$

$$\boxed{\|y(t)\| \leq \|x_1(t_0)\| + \int_{t_0}^{t}\|u(\tau)\|\,d\tau}$$

→ Bounded i/p will give you bdd o/p. But $x_2$ blows up.

⇒ Introduce some basic notions and results of External Stability and connect them to internal ⇒ Function spaces/Causality/Fb/WP/Passive

→ Cannot have infinite energy over an infinite tim

$\xi : [0,\infty) \to \mathbb{R}^n$

- Truncation operator: $(\cdot)_T$, $T \geq 0$

$$(\xi)_T(t) = \begin{cases} \xi(t) & t \leq T \\ 0 & t > T \end{cases}$$

- Lp Space: $L_p^n = \left\{ \xi : [0,\infty) \to \mathbb{R}^n \,\middle|\, \int_0^\infty \|\xi(t)\|^p dt < \infty \right\}$

$\sqrt[p]{\int \|\xi\|^p dt} = \|\xi\|_p$

- Extended Lp Space:

$\boxed{L_p \subset L_{pe}}$

$$L_{pe}^n = \left\{ \xi : [0,\infty) \to \mathbb{R}^n \,\middle|\, (\xi)_T \in L_p \;\forall T \geq 0 \right\}$$

- $\xi(t) = t$ : $\xi \in L_{pe}^1$ $\xi \notin L_p^1$ $p \in [1,\infty)$

- $\xi(t) = \begin{cases} 0 & t = 0 \\ 1/t & t > 0 \end{cases}$ $\xi \in L_{pe}^1$ $\xi \notin L_p^1$

We need extended Lp so that the integrals conv for trunc' fn

- $\forall p \in [1,\infty]$, $L_{pe}$ is linear. If $\xi \in L_{pe}^n$ then

— $\|\xi_T(\cdot)\|$ is a non-dec fn. of $T$.

— $\xi \in L_p^n$ iff $\exists\, m > 0$ s.t. $\|\xi_T\| \leq m$ $\forall T \geq 0$

In that case, $\|\xi\|_p = \lim_{T\to\infty} \|\xi_T\|$

Causal $F : L_{pe}^m \to L_{pe}^q$ is said to be causal ~~system~~ map if

$$(F(u))_T = (F(u_T))_T \quad \forall T \geq 0 \quad \forall u \in L_{pe}^m$$

→ **Causality** : Granger / $\Delta$MI / transfer entropy

1969 / N 2003
July
Moon Landing

$F : L_{pe}^{m} \to L_{pe}^{r}$ is causal iff $\forall u_1, u_2 \in L_{pe}^{m}$,

$$(u_1)_T = (u_2)_T \text{ for some } T > 0 \Rightarrow (F(u_1))_T = (F(u_2))_T$$

**Stability** : Finite Gain L-stability:

$F : L_{pe}^{m} \to L_{pe}^{r}$ is stable if $\exists\ \gamma, \beta > 0$ s.t. ← finite const.

$$\boxed{\|(F(u))_T\| \leq \gamma \|u_T\| + \beta}$$

$\beta$ ← OFFSET

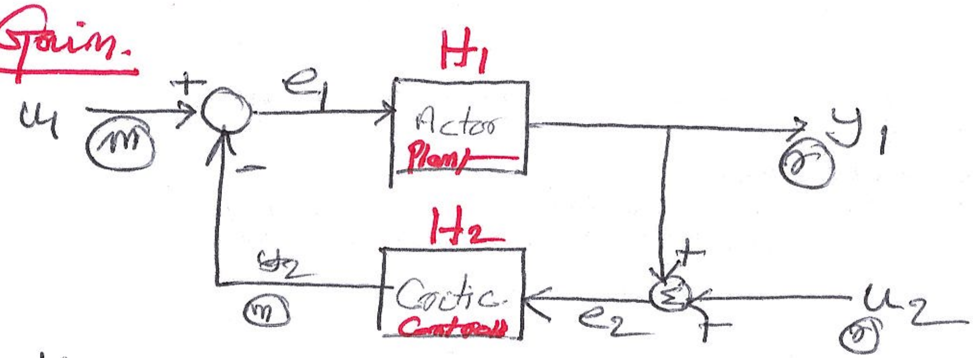
smallest such $\gamma$ is called GAIN

Alternatively F is stable if

(i) $u \in L_p^{m} \Rightarrow F(u) \in L_p^{r}$

(ii) $\exists$ finite const. $\gamma, \beta$ s.t. $\|F(u)\| \leq \gamma \|u\| + \beta \quad \forall u \in L_p^{m}$

**Small Gain.**




Example
set point
Watts governor

(i) $H_1, H_2$ are causal

(ii) $H_i$'s are stable w/ $\gamma_i$'s and $\beta_i$'s

(iii) $\forall u_1, u_2\ \exists$ unique outputs $e_1$ & $e_2$. Well Posed.

Lect 16

Lyap

Then if $\boxed{\alpha_1 \alpha_2 < 1}$

(i) $\|(e_1)_T\| \leq \left(\frac{1}{1-\alpha_1\alpha_2}\right)\left[\|(u_1)_T\| + \alpha_2\|(u_2)_T\| + \beta_2 + \alpha_2\beta_1\right]$

$\|(e_2)_T\| \leq \left(\frac{1}{1-\alpha_1\alpha_2}\right)\left[\|(u_2)_T\| + \alpha_1\|(u_1)_T\| + \beta_1 + \alpha_1\beta_2\right]$

$\forall\, T \geq 0$ for any $u_1, u_2 \in L_{pe}$

(ii) If $u_1, u_2 \in L_p$ the $(e_1, y_2)$ $(e_2, y_1)$ $L_p$ and their norms are bounded a/o true.

Interp. FB system is stable (FGLS) if $\alpha_1 \alpha_2 < 1$.

(iii) is hard to verify $\gg$ stronger assumption.

$F: L_{pe}^m \to L_{pe}^r$ is incrementall FGS if

(i) $\boxed{F(0) \in L_{pe}^{q}}$ (can be related to Lyap stable)

(ii) $\forall\, T > 0,\ u, v \in L_{pe}^m \ \exists\ k > 0$ indep $T, u, v$.

$\|(F(u_T))_T - (F(v))_T\| \leq k\|u_T - v_T\|$

(III) $F: L_{pe}^m \rightarrow L_{pe}^r$ is IFGS with gain $K < 1$.

Then, $\exists$ unique $u^* \in L_{pe}^m$

$$F(u^*) = u^*$$

(IIII)

$$H_1(u)(t) = \int_0^t \exp(-a(t-\sigma))\, u(\sigma)\, d\sigma$$

$$H_2(u)(u) = k\, u(t) \qquad a > 0$$

$$\alpha_1 = \frac{1}{a} \quad \beta_1 = 0$$

$$\alpha_2 = |k| \quad \beta_2 = 0 \quad \rightarrow \frac{|k|}{a} < 1$$

$$\rightarrow |k| < a$$

$$\boxed{-a < k < a}$$ conservative.

$k > -a$ is $m+s$ condn.

$$N(u) \leq \|N(u) - N(0)\| + \|N(0)\|$$

Passive.

Diss inequality

$$\boxed{\frac{dH}{dt} \leq \langle q, f \rangle}$$

$$\dot{q} = \frac{\partial H}{\partial p} \rightarrow o/p$$

$$\dot{p} = -\frac{\partial H}{\partial q} + \theta \quad i/p$$

optimized Lagrange

grand disc.

Hamiltonian.

$$\boxed{\dot{x} = [J(x) - D(x)] \nabla_x H + B(x) u}$$

$$y = B^T \nabla_x H$$

$$\boxed{\dot{H} = \langle \nabla_x H, \dot{x} \rangle = -\nabla H^T D \nabla H + \nabla H B(x) u}$$