Remember that a mechanical system without friction or an electrical system without resistance can be expressed in the Hamiltonian form.

$$\boxed{H = \frac{1}{2}Kx^2 + \frac{1}{2}M\dot{x}^2} \qquad \boxed{H = \frac{1}{2C}q^2 + \frac{1}{2}L\dot{q}^2}$$

$$(x, M\dot{x}) \leftarrow (p,q) \rightarrow (q, L\dot{q})$$

$$\boxed{\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p} \\ \dot{p} &= -\frac{\partial H}{\partial q} + f \end{aligned}} \quad (1)$$

q: Generalized Coordinate/Position

p: Generalized Momentum

f: Generalized Force (Effort)

→ Force/Torque/Voltage etc

→ Then we can define the rate at which electrical/mechanical work is being done.

$$\boxed{S = \langle f, \dot{q} \rangle = f^T \dot{q}}$$

It is also known as the Supply Rate.

Then, time rate of change of the Hamiltonian H can be expressed as —

$$\begin{aligned}
\dot{H} &= \left(\frac{\partial H}{\partial q}\right)^T \dot{q} + \left(\frac{\partial H}{\partial p}\right)^T \dot{p} \\
&= \left(\frac{\partial H}{\partial q}\right)^T \left(\frac{\partial H}{\partial p}\right) - \left(\frac{\partial H}{\partial p}\right)^T \left(\frac{\partial H}{\partial q}\right) + \left(\frac{\partial H}{\partial p}\right)^T f \\
&= \dot{q}^T f = S
\end{aligned}$$

11/21/2017
BD | 18-2

Thus the stored energy of the system (H) changes at a rate same as the supply rate S. One can interpret the supply rate as the power input to the system.

In general if there is any dissipation (i.e loss of stored energy) due to damping/friction/resistance, we have the following relationship-

$$\boxed{\frac{dH}{dt} \leq S.} \qquad (2)$$

Then, by treating the generalized forces and velocities as inputs and outputs respectively, we have the following (weaker) version of (2), known as the dissipation inequality:

$$\boxed{H(q(t), p(t)) \leq H(q(0), p(0)) + \int_0^t u^T(\tau)\, y(\tau)\, d\tau} \qquad (3)$$

where,
$u \triangleq f$ and $y \triangleq \dot{q}$.

Moreover, if the Hamiltonian is bounded below, i.e. $H(q,p) \geq C$ for every q, p, (3) can be expressed as —

$$\begin{aligned}
\int_0^t u^T(\tau)\, y(\tau)\, d\tau &\geq H(q(t), p(t)) - H(q(0), p(0)) \\
&\geq C - H(q(0), p(0)) \\
&= -\delta
\end{aligned}$$

where $\delta \triangleq H(q(0), p(0)) - C \geq 0$ depends on the initial condition.

Passivity and Dissipative Systems:—

In what follows we will see a further generalization of this concept. Here we consider nonlinear systems with dynamics given by —

$$\boxed{\begin{aligned} \dot{x} &= f(x,u) \\ y &= h(x,u) \end{aligned}} \qquad (4)$$

where, $x(t) \in \mathbb{R}^n$, $u(t) \in U \subset \mathbb{R}^m$ and $y(t) \in Y \subseteq \mathbb{R}^p$ at every $t \geq 0$, together with a function

$$\boxed{\begin{aligned} s &: U \times Y \longrightarrow \mathbb{R} \\ &(u(t), y(t)) \longmapsto s(u(t), y(t)), \end{aligned}} \qquad (5)$$

which we call the <u>supply rate</u> of (4).

→ System (4) is called to be <u>dissipative with respect to the supply rate</u> s if there exists a function $\Phi : \mathbb{R}^n \rightarrow [0, \infty)$, called the <u>storage function</u>, such that for all $t_1 \geq t_0$, any $x(t_0) \in \mathbb{R}^n$ and any input funcion $u(\cdot)$, we have —

$$\boxed{\Phi(x(t_1)) \leq \Phi(x(t_0)) + \int_{t_0}^{t_1} s(u(\sigma), y(\sigma))\, d\sigma} \qquad (6)$$

where $x(t_1)$ is the state of (4) at time t_1 resulting from the initial condition $x(t_0)$ at time t_0 under the input $u(\cdot)$. Condition (6) is the more general form of dissipation inequality.

→ As discussed in the last class, a nonlinear system can also be viewed as a function or a map between two appropriate function spaces; more specifically between extended L_2-spaces.

11/21/2017
BD (18-4)

- Consider, $G : L_{2e} \to L_{2e}$. Then, G is <u>passive</u> if there exists some constant $\beta \in \mathbb{R}$ such that —

$$\boxed{\int_0^t u^T(\tau)\big(G(u)\big)(t)\, d\tau \geq \beta}$$

for every $u \in L_{2e}$ and $t \geq 0$. Moreover, G is <u>strictly input passive</u> if there exists $\beta \in \mathbb{R}$ and $\gamma > 0$ such that —

$$\boxed{\int_0^t u^T(\tau)\big(G(u)\big)(\tau)\, d\tau \geq \gamma \int_0^t u^T(\tau)\, u(\tau)\, d\tau + \beta}$$

for every $u \in L_{2e}$ and $t \geq 0$. G is <u>strictly output passive</u> if there exist $\beta \in \mathbb{R}$ and $\gamma > 0$ such that,

$$\boxed{\int_0^t u^T(\tau)\big(G(u)\big)(\tau)\, d\tau \geq \gamma \int_0^t \big(G(u)\big)^T(\tau)\big(G(u)\big)(\tau)\, d\tau + \beta}$$

for every $u \in L_{2e}$ and $t \geq 0$.

- Then we can say that the system (4) is <u>passive</u> if it is dissipative with respect to the supply rate —

$$s\big(u(t), y(t)\big) = u^T(t)\, y(t).$$

Moreover, (4) is <u>strictly input passive</u> if there exists $\delta > 0$ such that (4) is dissipative with respect to —

$$s\big(u(t), y(t)\big) = u^T(t)\, y(t) - \delta \|u(t)\|_2^2.$$

Finally (4) is <u>strictly output passive</u> if it is dissipative with respect to —

$$s\big(u(t), y(t)\big) = u^T(t)\, y(t) - \delta \|y(t)\|_2^2 \text{ where } \delta > 0$$

11/21/2017
BD (18-5)

- The system (4) is finite gain L_2-stable with gain $\leq \gamma$ if it is dissipative with respect to —

$$s(u(t), y(t)) = \frac{1}{2}\gamma^2 \|u(t)\|_2^2 - \|y\|_2^2.$$

From dissipation inequality —

$$\frac{\gamma^2}{2}\int_0^t u^T(\tau)u(\tau)\,d\tau - \int_0^t y^T(\tau)y(\tau)\,d\tau \geq \Phi(x(t)) - \Phi(x(0))$$
$$\geq -\Phi(x(0))$$

This can be expressed as —

$$\int_0^t y^T(\tau)y(\tau)\,d\tau \leq \frac{\gamma^2}{2}\int_0^t u^T(\tau)u(\tau)\,d\tau + \Phi(x(0))$$

- Also, system (4) is finite gain L_2-stable if it is strictly output passive.

$$\int_0^t u^T(\tau)y(\tau)\,d\tau \geq \gamma\int_0^t y^T(\tau)y(\tau)\,d\tau + \beta \qquad \gamma > 0, \beta \in \mathbb{R}$$

Therefore,

$$\int_0^t y^T(\tau)y(\tau)\,d\tau \leq \frac{1}{\gamma}\int_0^t u^T(\tau)y(\tau)\,d\tau - \beta/\gamma$$

$$\leq \frac{1}{\gamma}\int_0^t u^T(\tau)y(\tau)\,d\tau - \left(\frac{\beta}{\gamma}\right) + \frac{1}{2\gamma}\int_0^t \left(\frac{u^T(\tau)}{\sqrt{\gamma}} - \sqrt{\gamma}\,y(\tau)\right)^T\left(\frac{u(\tau)}{\sqrt{\gamma}} - \sqrt{\gamma}\,y(\tau)\right)d\tau$$

$$= -\frac{\beta}{\gamma} + \frac{1}{2\gamma}\left(\frac{1}{\gamma}\int_0^t u^T(\tau)u(\tau)\,d\tau + \gamma\int_0^t y^T(\tau)y(\tau)\,d\tau\right)$$

Hence,

$$\int_0^t y^T(\tau)y(\tau)\,d\tau \leq \frac{1}{\gamma^2}\int_0^t u^T(\tau)u(\tau)\,d\tau - \frac{2\beta}{\gamma}$$

- Then, by restricting our focus to infinitesimal movements along trajectories of (4), we can express the dissipation inequality as —

$$\boxed{\left(\frac{\partial \Phi}{\partial x}\right)^T f(x,u) \leq s(u,y)} \tag{7}$$

This inequality is called the <u>differential dissipation inequality</u>.

- Now assume $\Phi : \mathbb{R}^n \to \mathbb{R}^+$ is a continuously differentiable (i.e C^1) storage function for the system defined by (4). Moreover, $x_* \in \mathbb{R}^n$ is a strict local minima of Φ, and the supply rate is such that

$$s(0,y) \leq 0 \text{ for all } y.$$

Then, x^* is a stable equilibrium of (4) when the external input is zero. To show this, define a Lyapunov function as —

$$V(x) = \Phi(x) - \Phi(x^*).$$

Clearly, $V(x^*) = 0$ and $V(x)$ is strictly positive for x in the neighborhood of x^*. Also,

$$\dot{V}(x) = \left(\frac{\partial \Phi}{\partial x}\right)^T f(x,0) \leq s(0,y) \leq 0,$$

from the differential dissipation inequality.

<u>INPUT-TO-STATE STABILITY (ISS)</u>:—

- Consider a function $\beta : [0,\infty) \times [0,\infty) \to [0,\infty)$. We call β to be a <u>class $K\mathcal{L}$-function</u> if $\beta(\cdot, t)$ is a class K_∞ function for each t and $\beta(x,t) \to 0$ as $t \to \infty$.

- The concept of input-to-~~stability~~ state stability (ISS) provides a means to merge the concepts of internal (Lyapunov) stability and external (e.g. Operator approach based input-output stability) stability. ISS also formalizes the concept of robustness with respect to disturbances. In what follows we consider the dynamics of the system to be governed by —

$$\boxed{\dot{x} = f(x, u)} \quad \text{——} \quad (8)$$

- This system (8) has the associated zero-system defined as —

$$\dot{x} = f(x, 0),$$

i.e. system with zero-input. Now, (8) is called 0-GAS if the associated zero system is globally asymptotically stable.

However 0-GAS does not necessarily mean that the system will yield a bounded output when excited by a bounded input. Consider —

$$\boxed{\dot{x} = -x + (x^2+1)u}.$$

Clearly the zero-system $\boxed{\dot{x} = -x}$ is GAS. But there are some bounded inputs (which even converges to zero) which lead to diverging solutions for the state.

Consider, $u(t) = 1/\sqrt{2t+2}$

and, $x(0) = \sqrt{2}$

Then, $x(t) = \sqrt{2t+2} \rightarrow \infty$ as $t \rightarrow \infty$.

Moreover, if the input is identically 1, i.e. $u \equiv 1$, we have —

$$\dot{x} = -x + x^2 + 1 = \left(x - \tfrac{1}{2}\right)^2 + \tfrac{3}{4} > 0$$

and, $$x(t) = \frac{\sqrt{3}}{2} \tan\left(\frac{\sqrt{3}}{2} t + \tan^{-1}\left(\frac{2x_0 - 1}{\sqrt{3}}\right)\right) + \frac{1}{2}$$

this solution explodes in finite time.

Thus we need more conditions on the dynamics to ensure that the states will be bounded under bounded input.

- The system $\dot{x} = f(x,u)$ (i.e. (8)) is <u>input-to-state stable</u> if there exist a <u>class $\mathcal{KL}$-function β</u> and a <u>class $\mathcal{K}$-function γ</u> such that —

$$\boxed{\|x(t)\| \leq \beta\left(\|x(t_0)\|, t - t_0\right) + \gamma\left(\sup_{t_0 \leq \tau \leq t} \|u(\tau)\|\right)}$$

for any initial condition $x(t_0)$ and any bounded input $u(t)$.

Then, we can claim that for a system which is ISS, the following hold true —

i) The state will always be bounded for bounded inputs.

ii) $x(t)$ is ultimately bounded by a class $\mathcal{K}$ function of $\sup_{t \geq t_0} \|u(t)\|$, i.e. solutions will eventually be trapped inside a ball of finite radius.

iii) Whenever $u(t) \to 0$ as $t \to \infty$, we have $x(t) \to 0$ as $t \to \infty$.

iv) The origin of $\dot{x} = f(x,0)$ [the zero-system] is GAS.

01/21/2017
BD (18-9)

- Let $V(x)$ be a continuously differentiable function such that —

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|)$$

and, $$\left(\frac{\partial V}{\partial x}\right)^T f(x,u) \leq -W(x) \quad \forall \|x\| \geq \rho(\|u\|) > 0$$

where $\alpha_1, \alpha_2 \in K_\infty$, $\rho \in K$ and W is a continuous positive definite function. Then, there system $\dot{x} = f(x,u)$ is ISS with $\gamma = \alpha_1^{-1} \circ \alpha_2 \circ \rho$.

- $\dot{x} = -x - 2x^3 + (x^2+1)u$

Let,

$$V(x) = \frac{1}{2}x^2$$

$$\dot{V}(x) = \left(\frac{\partial V}{\partial x}\right) f(x,u)$$

$$= -x^2 - 2x^4 + (x^2+1)xu$$

$$= -x^4 - x^2(x^2+1) + xu(x^2+1)$$

$$= -x^4 - (x^2+1)\left(x(x-u)\right)$$

$$\leq -x^4 \quad \text{when } |x| \geq |u|$$

Therefore this system is ISS with —

$$\gamma(r) = r$$