Lecture 19

MAE 544
"NONLINEAR CONTROL"

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A recurring problem in nonlinear control is to find a smooth feedback law

$$\boxed{u = u(x) = \alpha(x)} \qquad x \in \mathbb{R}^n,\ \alpha: \mathbb{R}^n \to \mathbb{R}$$

such that $x = x^*$ is a globally asymptotically stable (G.A.S.) equilibrium point for the system.

$$\boxed{\dot{x} = f(x, u)} \qquad f: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \qquad (1)$$

where, $f(x^*, 0) = 0$. As discussed earlier we assume x^* to be origin (i.e. $x^* = 0$) without loss of generality. Then our objective can be restated as finding out a smooth feedback $\alpha: \mathbb{R}^n \to \mathbb{R}$ such that $x = 0$ is G.A.S. (or simply asymptotically stable, in a more relaxed setting) for the closed loop dynamics —

$$\boxed{\dot{x} = f(x, \alpha(x))}$$

Then by converse Lyapunov theorem, we can conclude that there exists a Lyapunov function $V(x)$ (i.e. positive definite) such that

$$\dot{V}(x) = \boxed{\left(\frac{\partial V}{\partial x}(x)\right)^T f(x, \alpha(x)) \leq -W(x)}$$

where $W(0) = 0$ and $W(x) > 0$ for all $x \neq 0$.
Such a V is called CONTROL LYAPUNOV FUNCTION for (1).

Control Lyapunov Function:—

Now we narrow our focus to a control-affine nonlinear system —

$$\boxed{\dot{x} = f(x) + g(x)u} \quad (2)$$

where, $x \in \mathbb{R}^n$, $u \in \mathbb{R}$ and $f(0) = 0$. Therefore $x^* = 0$ is an equilibrium of (2). Suppose, $u = \alpha(x)$ is a continuous state feedback such $x^* = 0$ is an asymptotically stable equilibrium of the closed-loop dynamics —

$$\boxed{\dot{x} = f(x) + g(x)\,\alpha(x).}$$

There, there exists a Lyapunov function $V(x)$ (i.e. locally positive definite) such that —

$$\left(\frac{\partial V}{\partial x}(x)\right)^T \Big(f(x) + g(x)\,\alpha(x)\Big) < 0 \quad \forall x \in D \setminus \{0\}$$

$$\Longleftrightarrow \boxed{L_f V(x) + \alpha(x) . L_g V(x) < 0} \quad \forall x \in D \setminus \{0\} \quad (3)$$

where $D \subset \mathbb{R}^n$ is some neighborhood around the origin.

- Moreover, if $u = \alpha(x)$ is a globally stabilizing continuous feedback, then it follows from converse Lyapunov theorem that —

  $D = \mathbb{R}^n$ and $V(x) \to \infty$ as $\|x\| \to \infty$.

- Now suppose, $L_g V(x) = 0$ for ~~every~~ some $x \in D \setminus \{0\}$.

Then, $L_f V(x) < 0$.

- As $\alpha(x)$ is continuous and $\alpha(0) = 0$, we can show that for every $\varepsilon > 0$, there exists a $\delta > 0$ such that ~~of~~ if $0 \leq \|x\| < \delta$, there is a u with $\|u\| < \varepsilon$ such that

$$\boxed{L_f V(x) + L_g V(x).\ u < 0.}$$

This ~~inequality~~ is known as the small control property.

→ A continuously differentiable locally positive definite function $V : \mathbb{R}^n \to \mathbb{R}_+$ is a control Lyapunov function (CLF) for the nonlinear system (2) if–

(i) It satisfies the small control property.

(ii) $L_f V(x) < 0$ ~~∀~~ $x \in D \setminus \{0\}$ whenever $L_g V(x) = 0$ ~~for~~ some $x \in D \setminus \{0\}$ where D is a neighborhood around the origin. In addition if V is radially unbounded and $D = \mathbb{R}^n$, then V is a global CLF.

- It directly follows from our discussion so far that if the system given by (2) is stabilizable by a continuous state feedback then it has a CLF. Arnstein - Sontag theorem shows that existence of a CLF is also a sufficient condition for stabilizablity.

→ THEOREM: Let $V(x)$ be a control Lyapunov function for (2). Then, origin is stabilizable by a continuous state feedback —

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$$\boxed{u = \alpha(x) = \begin{cases} -\dfrac{L_fV(x) + \sqrt{(L_fV(x))^2 + (L_gV(x))^4}}{L_gV(x)} & \text{if } L_gV(x) \neq 0 \\ 0 & \text{if } L_gV(x) = 0 \end{cases}} \quad (4)$$

defined ~~over~~ continuously over a neighborhood D_0 around the origin. Moreover, if f and g are smooth then α is smooth whenever $x \neq 0$. Also, the state-feedback $u = \alpha(x)$ is globally stabilizing if V is a global CLF.

— Sketch of proof:

Continuity and almost smoothness of $\alpha(x)$ can be shown by exploiting properties of f and g.

Now, if $L_gV(x) = 0$,

$$\dot{V}(x) = L_fV(x) < 0 \quad \text{whenever } x \neq 0.$$

On the other hand, if $L_gV(x) \neq 0$,

$$\begin{aligned} \dot{V}(x) &= L_fV(x) + \alpha(x) \cdot L_gV(x) \\ &= L_fV(x) - L_fV(x) - \sqrt{(L_fV(x))^2 + (L_gV(x))^4} \\ &= -\sqrt{(L_fV(x))^2 + (L_gV(x))^4} \\ &< 0 \quad \text{whenever } x \neq 0. \end{aligned}$$

- But it still remains a challenge to find a CLF for a given system. However if we know of any stabilizing control with a corresponding Lyapunov function $V(x)$, then $V(x)$ is a CLF.

An Example:—

Consider the scalar nonlinear system —

$$\boxed{\dot{x} = f(x) + u\,g(x)}$$

Let, $V(x) \triangleq \frac{1}{2}x^2$

continuously differentiable; positive definite.

Moreover, we assume if $x \neq 0$ and $g(x) = 0$ then $xf(x) < 0$.

Then we can also show that the small control property holds true, and hence by using Sontag's formula —

$$u = \alpha(x) = \begin{cases} -\dfrac{x f(x) + |x|\sqrt{(f(x))^2 + x^2 (g(x))^4}}{x g(x)} & \text{when } xg(x) \neq 0 \\ 0 & \text{when } xg(x) = 0 \end{cases}$$

With this stabilizing feedback, we can express the closed loop dynamics as —

$$\dot{x} = f(x) - \left[f(x) + \mathrm{sgn}(x)\sqrt{(f(x))^2 + x^2(g(x))^4}\right]$$

$$\Rightarrow \boxed{\dot{x} = -\mathrm{sgn}(x)\sqrt{(f(x))^2 + x^2(g(x))^4}}$$

BACKSTEPPING:—

Backstepping provides an algorithmic approach to design control Lyapunov functions (CLF) for a given system, together with the corresponding stabilizing feedback law. In doing so, it leverages the insight and simplicity from a scalar design process.

Integrator Backstepping:—

Consider the system —

$$\boxed{\begin{aligned}\dot{x} &= f(x) + \xi\, g(x)\\ \dot{\xi} &= u\end{aligned}} \qquad (5)$$

where, $x \in \mathbb{R}^n$, $\xi \in \mathbb{R}$, $u \in \mathbb{R}^{m}$ and f, g are smooth in a neighborhood (D) of $x = 0$. Moreover, $f(0) = 0$.

Our objective is design a state feedback $u = \alpha(x, \xi)$ which will stabilize the origin $(x, \xi) = (0, 0)$.

Now we assume that $\boxed{\xi = \phi(x)}$ stabilizes the origin of —

$$\boxed{\dot{x} = f(x) + \phi(x) g(x)} \qquad (6)$$

We also assume that $V : \mathbb{R}^n \to \mathbb{R}_+$ is a Lyapunov function which establishes stability of $x = 0$ for (5), i.e. V is L.P.D.F. and

$$\dot{V} = \left(\frac{\partial V}{\partial x}(x)\right)^T \left(f(x) + \phi(x) g(x)\right) \leq -W(x) \quad \forall x \in D$$

where, $W(x) > 0 \ \forall x \in D \setminus \{0\}$ and $W(0) = 0$.

Then by introducing $z \triangleq \xi - \phi(x)$, we can rewrite (5) as —

$$\dot{x} = \left[f(x) + \phi(x) g(x)\right] + z \cdot g(x)$$

$$\dot{z} = u - \left(\frac{\partial \phi}{\partial x}(x)\right)^T \left[\left(f(x) + \phi(x) g(x)\right) + z \cdot g(x)\right]$$

$$\dot{z} \triangleq v$$

This is very similar to the original dynamics (5), except the fact that now $x=0$ is an asymptotically stable equilibrium for the first equation whenever $z=0$.

Now we define a new Lyapunov function —

$$\tilde{V} : \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}_+$$

$$(x, z) \longmapsto V(x) + \frac{1}{2}(\xi - \phi(x))^2$$

$$= V(x) + \frac{1}{2} z^2$$

Clearly $\tilde{V}$ is L.P.D.F.

Moreover, $$\dot{\tilde{V}} = \left(\frac{\partial \tilde{V}}{\partial x}(x,z)\right)^T \dot{x} + \left(\frac{\partial \tilde{V}}{\partial z}(x,z)\right)^T \dot{z}$$

$$= \left(\frac{\partial V}{\partial x}(x)\right)^T \left[f(x) + \phi(x) g(x)\right] + \left(\frac{\partial V}{\partial x}(x)\right)^T g(x) z + z v$$

$$\leq -W(x) + z\left[v + \left(\frac{\partial V}{\partial x}(x)\right)^T g(x)\right]$$

Then by choosing v as

$$\boxed{v = -kz - \left(\frac{\partial V}{\partial x}(x)\right)^T g(x)}, \quad k>0$$

we have,

$$\dot{\tilde{V}} \leq -W(x) - kz^2$$

So, $(x, z) = (0,0)$ is asymptotically stable, and in addition if $\phi(0) = 0$, then $(x, \xi) = (0,0)$ is also asymptotically stable. with —

$$\boxed{u = \alpha(x,\xi) = -k(\xi - \phi(x)) - L_g V(x) + L_f \phi(x) + \xi . L_g \phi(x)} \quad (7)$$

In addition if all assumptions hold globally and $V(x)$ is radially unboundded then (7) will result in global asymptotic stabilization of $(x, \xi) = (0,0)$ for the closed-loop dynamics of (5).

The primary intuition behind integrator backstepping is to treat ξ as an input to the dynamics of x, and ~~then~~ design a controller which will stabilize ξ at ~~an~~ stabilizing input for x-dynamics.

→ An example:

$$\boxed{\begin{aligned} \dot{x}_1 &= x_1^2 - x_1^3 + x_2 \\ \dot{x}_2 &= u \end{aligned}}$$

Here, $f(x_1) = x_1^2 - x_1^3$, $g(x_1) = 1$ and we can easily show $x_2 = \phi(x_1) = -x_1^2 - x_1$ makes $x_1 = 0$ G.A.S. for the x_1-dynamics. (i.e. $\dot{x}_1 = -x_1 - x_1^3$).
~~The~~ A corresponding Lyapunov function is given by $V(x_1) = \frac{1}{2} x_1^2$.

$$\dot{V}(x_1) = -x_1^2 - x_1^4 < 0 \quad \forall\, x_1 \in \mathbb{R} \setminus \{0\}$$

and, $\dot{V}(0) = 0$.

Then we have the overall Lyapunov function —

$$\tilde{V}(x_1, x_2) = \frac{1}{2}\left[x_1^2 + (x_2 + x_1 + x_1^2)^2\right]$$

and, the stabilizing feedback —

$$u = \alpha(x_1, x_2) = -k(x_2 + x_1 + x_1^2) - x_1 - (2x_1 + 1)(x_1^2 - x_1^3 + x_2)$$

Generalization and Extension to Strict Feedback Form:—

$$\boxed{\begin{aligned}\dot{x} &= f_0(x) + g_0(x)\xi \\ \dot{\xi} &= f_1(x,\xi) + g_1(x,\xi)u\end{aligned}} \qquad x \in \mathbb{R}^n,\ \xi \in \mathbb{R},\ u \in \mathbb{R} \qquad (8)$$

where,

f_i and g_i are smooth.

If $g_1(x,\xi) \neq 0$ over the domain of our interest then by defining u_1 as —

$$u_0 = \frac{1}{g_1(x,\xi)}\left[u_1 - f_1(x,\xi)\right],$$

(8) can be expressed in the previously discussed form (5). Then u_1 can be solved by integrator backstepping, and that will yield u.

→ Strict Feedback Form:—

$$\boxed{\begin{aligned}\dot{x} &= f_0(x) + g_0(x)\xi_1 \\ \dot{\xi}_1 &= f_1(x,\xi_1) + g_1(x,\xi_1)\xi_2 \\ \dot{\xi}_2 &= f_2(x,\xi_1,\xi_2) + g_2(x,\xi_1,\xi_2)\xi_3 \\ &\ \vdots \\ \dot{\xi}_k &= f_k(x,\xi_1,\dots,\xi_k) + g_k(x,\xi_1,\dots,\xi_k)u\end{aligned}} \qquad (9)$$

where, $x \in \mathbb{R}^n$, $\xi_i \in \mathbb{R}$, $u \in \mathbb{R}$ and f_i, g_i are smooth.

We also assume, $g_i(x,\xi_1,\dots,\xi_i) \neq 0$, $1 \leq i \leq k$ over the domain of our interest. Then apply the treatment for (8) in a recursive way.