Passivity Based Control:—

The key idea is to make the closed loop system passive (for the given pair of inputs and outputs) with respect to some appropriate storage function which has a minimum at the desired equilibrium. In addition, if we impose detectability/observability of the associated output, we have asymptotic stability of the equilibrium.

→ Consider the system —

$$\boxed{\begin{aligned} \dot{x} &= f(x, u) \\ y &= h(x). \end{aligned}} \qquad x \in \mathbb{R}^n;\ u, y \in \mathbb{R}^m \quad \text{—— (1)}$$

This system is called zero-state observable if $u(t) \equiv 0$ and $y(t) \equiv 0$ implies $x(t) \equiv 0$.

For example, the system —

$$\dot{x}_1 = x_2\ ;\quad \dot{x}_2 = -x_1^3 + u\ ;\quad y = x_2$$

is zero-state observable.

Now we assume that $x = 0$ is an equilibrium for (1) (i.e. $f(0,0) = 0$) and this system is passive with a storage function V which is positive definite and radially unbounded.

If we assume that (1) is zero-state observable as well, then we can show that the origin ($x=0$) can be globally stabilized by an output feedback —

$$u = -\phi(y)$$

where $\phi: \mathbb{R}^m \to \mathbb{R}^m$ is such that $\phi(0)=0$ and $y^T\phi(y) > 0$ whenever $y \neq 0$.

$$\dot{V} = \left(\frac{\partial V}{\partial x}(x)\right)^T f(x, -\phi(y)) \underset{\text{from passivity}}{\leq} y^T(-\phi(y)) \underset{\text{from prop. of } \phi}{\leq} 0 \begin{cases} < 0 & \forall y \neq 0 \\ = 0 & y = 0 \end{cases}$$

Clearly $\dot{V}(x(t)) \equiv 0 \Rightarrow y(t) \equiv 0$

$$\begin{array}{c} y(t) \equiv 0 \\ \Downarrow \\ u(t) \equiv 0 \end{array} \Rightarrow x(t) \equiv 0$$

zero state observability

→ Now we shift our focus to an input-affine system —

$$\boxed{\dot{x} = f(x) + g(x)u}$$

where, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and the output is not defined. We also assume that there is a positive definite function V such that

$$L_f V(x) \leq 0 \quad \forall x \in \mathbb{R}^n.$$

If we define an output as —

$$y = h(x) \triangleq \left[\left(\frac{\partial V}{\partial x}(x)\right)^T g(x)\right]^T = g^T(x)\frac{\partial V}{\partial x}(x),$$

$$\dot{V} = L_f V(x) + \left(\frac{\partial V}{\partial x}(x)\right)^T g(x)u \leq y^T u$$

For example, consider the system —

$$\dot{x}_1 = x_2 \quad ; \quad \dot{x}_2 = -x_1^3 + u.$$

By defining $V(x) \triangleq \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2$, we have —

$$L_f V(x) = x_1^3 x_2 - x_2 x_1^3 = 0.$$

Then, we choose an output as —

$$y = h(x) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1^3 \\ x_2 \end{bmatrix} = x_2$$

Thus, this system is zero-state observable and passive with respect to V. $\boxed{u = -k\,y}$ will globally stabilize the origin.

→ In addition to making a system passive by choosing an appropriate output, one can also introduce passivity by designing a suitable feedback.

The system

$$\boxed{\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned}}$$

is equivalent to a passive system if there is a state feedback —

$$u = \alpha(x) + \beta(x) v$$

such that —

$$\dot{x} = \left[f(x) + g(x)\alpha(x) \right] + \left[g(x)\beta(x) \right] v$$
$$y = h(x)$$

is passive with some storage function.

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BD | 20-4

→ A Large class of mechanical systems (e.g. robotic manipulators / inverted pendulum) can modelled as —

$$\boxed{\underbrace{M(q)\ddot{q}}_{\text{Inertial}} + \underbrace{C(q,\dot{q})\dot{q}}_{\text{Coriolis/Centrifugal}} + \underbrace{D\dot{q}}_{\text{Dissipation/Friction}} + \underbrace{g(q)}_{\text{potential/gravitational}} = \underbrace{u}_{\text{External}}}$$

In addition, we have —

- $M = M^T > 0$
- $\dot{M} - 2C$ is skew-symmetric, and
- $D = D^T \geq 0$

Objective: Stabilize the system at q_r (desired configuration)

By defining $e \triangleq q - q_r$, we have $\dot{e} = \dot{q}$ and $\ddot{e} = \ddot{q}$. As a consequence the dynamics can be expressed as —

$$M(q)\ddot{e} + C(q,\underbrace{\dot{q}}_{=\dot{e}})\dot{e} + D\dot{e} + g(\underbrace{q}_{=e+q_r}) = u$$

$(e=0, \dot{e}=0)$ is not an open loop equilibrium point.

Define,

$$u = g(q) - \phi(e) + v$$

where, $\phi(0) = 0$ and $e^T\phi(e) > 0 \;\; \forall e \neq 0$.

Then, $M(q)\ddot{e} + C(q,\dot{q})\dot{e} + D\dot{e} + \phi(e) = v$

By letting V be —

$$V \triangleq \frac{1}{2}\dot{e}^T M(q)\dot{e} + \int_0^e \phi^T(\tau)\,d\tau > 0 \quad \forall (e,\dot{e}) \neq (0,0)$$

we have ——

$$\dot{V} = \frac{1}{2}\dot{e}(\dot{M} - 2C)\dot{e} - \dot{e}^T D\dot{e} - \dot{e}^T\phi(e) + \dot{e}^T v + \phi^T(e)\dot{e}$$

$$\leq \dot{e}^T v$$

Therefore this system is passive with ~~the~~ $\dot{e}$ as an output. We can also show that it is zero state observable. This means that we can design an output feedback

$$v = -\phi_d(\dot{e}) \qquad \phi_d(0) = 0,\ \dot{e}^T\phi_d(\dot{e}) > 0 \ \forall \dot{e} \neq 0$$

which will stabilize the origin.

$$u = g(q) - \phi(e) - \phi_d(\dot{e})$$

can lead to PD control.

Brief Intro. to Controlled Lagrangian :—

This allows us to design stabilizing control laws for Lagrangian systems with the Lagrangian defined as the difference of kinetic and potential energy. The key idea is to consider control laws which result in a Lagrangian form for the closed loop dynamics. In particular we get control laws as a result of the energy modification.

→ Lagrangian Systems:—

$q = (q^1, \text{---}, q^n)$: generalized coordinates

$K = \frac{1}{2} \dot{q}^T g(q) \dot{q}$ ← kinetic energy

$g(q)$ ← inertia tensor/metric

$= \frac{1}{2} \dot{q} \cdot g(q) \dot{q}$

$= \frac{1}{2} g_{ab} \dot{q}^a \dot{q}^b$ ← Einstein notation

$V = V(q)$ ← Potential energy.

Lagrangian : $L = K - V$

$$= \frac{1}{2} g_{ab} \dot{q}^a \dot{q}^b - V(q^a)$$

Euler Lagrange equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = Q_i$$ ← generalized force.

Total energy $(E) = K + V$

Clearly, if $(q, \dot{q}) = (q_e, 0)$ is an equilibrium of this Lagrangian system then q_e must be a critical point of V. Moreover, by Lagrange-Dirichlet theorem, we can show that this equilibrium is stable if the second variation of E (i.e. the matrix $\delta^2 E$ of size $2n \times 2n$) is definite at $(q_e, 0)$.