Lagrangian Control Systems:—

For a given system assume that the generalized coordinates can be groups into two categories—

$$q = (x^\alpha, \theta^a) \qquad \alpha = 1, -, k \; ; \; a = 1, --, r$$

such that θ^a directions have access to external control inputs. This framework allows us to consider underactuated systems.

$$L = L(x^\alpha, \dot{x}^\beta, \theta^a, \dot{\theta}^b)$$

$$= \frac{1}{2} g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta + g_{\alpha a} \dot{x}^\alpha \dot{\theta}^a + \frac{1}{2} g_{ab} \dot{\theta}^a \dot{\theta}^b - V(x^\alpha, \theta^a)$$

Then the open-loop dynamics are given by—

$$\boxed{\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^\alpha}\right) - \frac{\partial L}{\partial x^\alpha} &= 0 \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}^a}\right) - \frac{\partial L}{\partial \theta^a} &= u_a \end{aligned}} \qquad (1)$$

Using controlled Lagrangian approach we aim to find u_a and a modified Lagrangian L_c such that (1) is equivalent to—

$$\boxed{\begin{aligned} \frac{d}{dt}\left(\frac{\partial L_c}{\partial \dot{x}^\alpha}\right) - \frac{\partial L_c}{\partial x^\alpha} &= 0 \\ \frac{d}{dt}\left(\frac{\partial L_c}{\partial \dot{\theta}^a}\right) - \frac{\partial L_c}{\partial \theta^a} &= 0 \end{aligned}} \qquad (2)$$

and the desired equilibrium $(x_e^\alpha, \theta_e^a, 0, 0)$ is stable for (2).

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Define,

$$\boxed{\begin{aligned} L_{c\ell} &= L(x^\alpha, \dot{x}^\beta, \theta^a, \dot{\theta}^b) + \frac{1}{2}\tau_{ab}\,\tilde{\tau}^a_\alpha \tilde{\tau}^b_\beta \dot{x}^\alpha \dot{x}^\beta \\ &+ \frac{1}{2}(\rho-1)\, g_{ab}\left(\dot{\theta}^a + g^{ac} g_{\alpha c}\dot{x}^\alpha + \tilde{\tau}^a_\alpha \dot{x}^\alpha\right)\left(\dot{\theta}^b + g^{bd} g_{\beta d}\dot{x}^\beta + \tilde{\tau}^b_\beta \dot{x}^\beta\right) \\ &- V_\varepsilon(x^\alpha, \theta^a) \end{aligned}} \quad (3)$$

where, $\tilde{\tau}^a_\alpha$ is a matrix of size $r \times k$,

τ_{ab} is matrix of size $r \times r$,

ρ is a constant scalar, and

V_ε is a modification of the potential energy.

Now assume following conditions hold true :—

SM-I : $\tau_{ab} = \tau\, g_{ab}$ (i.e. τ_{ab} is a scalar multiple of the inertia metric in θ^a directions)

SM-II : g_{ab} is independent of x^α (i.e. partial invariance of the metric tensor).

SM-III : $\tilde{\tau}^b_\alpha = \left(-\frac{1}{\tau}\right) g^{ab} g_{\alpha a}$

SM-IV : $\frac{\partial g_{\alpha a}}{\partial x^\delta} = \frac{\partial g_{\delta a}}{\partial x^\alpha}$

SM-V : $\frac{\partial^2 V}{\partial x^\alpha \partial \theta^a} g^{ad} g_{\beta d} = \frac{\partial^2 V}{\partial x^\beta \partial \theta^a} g^{ad} g_{\alpha d}$

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Condition SM-III and SM-IV imply that the given system should lack gyroscopic forces. Also, SM-V ensures that the choice of V_ε is legitimate.

Main result: Under these 5 assumptions (SM-I to SM-V), the Euler-Lagrange equations for L_c (i.e. (2) together with (3)) coincides with the Euler-Lagrange equations for L (1) with the following u_a —

$$\boxed{u_a = -\frac{d}{dt}\left(g_{ab}\,\tau^b_\alpha\,\dot{x}^\alpha\right) + \left(\frac{\rho-1}{\rho}\right)\frac{\partial V}{\partial \theta^a} - \frac{1}{\rho}\frac{\partial V_\varepsilon}{\partial \theta^a}} \quad (4)$$

The associated equilibrium is Lyapunov stable if it is a critical point of $(V+V_\varepsilon)$ and the second derivative of the total energy (E_c corresponding to L_c) is definite at that point.

But we are yet to achieve asymptotic stability for which we will add a dissipative term into the control u_a. SM-III and SM-IV ensure that there exists a function $h:\mathbb{R}^k \to \mathbb{R}^r$ which locally satisfies:

$$\frac{\partial h^a}{\partial x^\alpha} = \left(\frac{\rho-1}{\rho} - \frac{1}{\sigma}\right) g^{ac}\, g_{\alpha c} \quad \text{and} \quad h^a(x_e) = 0$$

Then, the dissipative correction term is given by —

$$\boxed{u_a^{diss} = C_a^d\, g_{bd}\left(\dot{\theta}^b + \frac{\partial h^b}{\partial x^\alpha}\dot{x}^\alpha\right)}$$

Here, C_a^d is a control gain matrix. This is chosen to be positive definite (resp. negative definite) if the equilibrium is a maximum (resp. minimum) of E_c.

Feedback Linearization:—

The key idea for feedback linearization is use coordinate transformation and a feedback so that the closed-loop dynamics in the transformed coordinate behave as a linear system.

Consider the nonlinear dynamics —

$$\dot{x} = f(x) + g(x)u = f(x) + \sum_{i=1}^{m} u_i g_i(x)$$

where, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and f, g_i's are smooth vector fields on $\mathbb{R}^n$.

Now, define a coordinate transformation $z = \Phi(x)$ and a feedback $u = \alpha(x) + \beta(x)v$.

Then, $$\dot{z} = \left(\frac{\partial \Phi}{\partial x}\right)_{x=\Phi^{-1}(z)} \left[f(\Phi^{-1}(z)) + g(\Phi^{-1}(z))\left[\alpha(\Phi^{-1}(z)) + \beta(\Phi^{-1}(z))v\right]\right].$$

Can we find Φ, α, β such that there exist $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ satisfying —

$$Az = \left[\frac{\partial \Phi}{\partial x}\left(f(x) + g(x)\alpha(x)\right)\right]\Bigg|_{x=\Phi^{-1}(z)}$$

and, $$B = \left[\frac{\partial \Phi}{\partial x}\left(g(x)\beta(x)\right)\right]\Bigg|_{x=\Phi^{-1}(z)}$$

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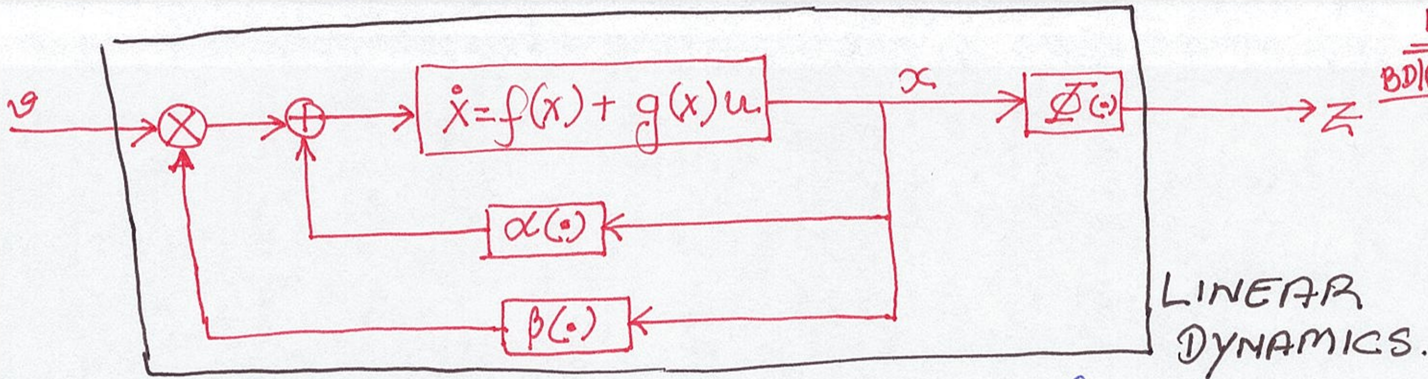

Then we can use tools from linear control theory to design controllers for this system.

As $z^*=0$ is an equilibrium for the linearized dynamics we have to choose the transformation in such a way that $x^*=\Phi^{-1}(0)$ is an equilibrium of the original dynamics, i.e. $f(x^*)=0$.

Feedback Linearization for a SISO system:-

Consider, $\boxed{\begin{aligned}\dot{x}&=f(x)+ug(x)\\ y&=h(x)\end{aligned}}$ ——— (5)

where, $x\in\mathbb{R}^n$; $u,y\in\mathbb{R}$; $f,g\in\mathfrak{X}(\mathbb{R}^n)$ and $h\in C^\infty(\mathbb{R}^n)$

Then, $\dot{y}=L_fh(x)+uL_gh(x)$

Suppose, $\exists\ K>0$ such that $|L_gh(x)|>K$ ~~for~~ every x in U, where U is an open neighborhood around an equilibrium x^* (i.e. $f(x^*)=0$)

Then, by choosing u as

$$u=\frac{1}{L_gh(x)}\left[v-L_fh(x)\right]\quad\forall x\in U$$

the dynamics of y become a single-integrator-

$$\dot{y}=v$$

The fact that $L_g h(x)$ is bounded away from zero (not only non-zero) ensures that the control does not become unbounded. However this linearized dynamics ($\dot{y} = v$) have rendered $(n-1)$ states unobservable.

Now assume,

$$L_g h(x) \equiv 0 \quad \forall x \in U.$$

Then we cannot define such a feedback. However in that case, we have

$$\dot{y} = L_f h(x)$$

which in turn leads to —

$$\ddot{y} = L_f^2 h(x) + u L_g L_f h(x)$$

Now, if $L_g L_f h(x)$ is bounded away from zero over an open neighborhood around x^*, we can define,

$$u = \frac{1}{L_g L_f h(x)}\left[v - L_f^2 h(x)\right],$$

and then the linearized dynamics will be —

$$\ddot{y} = v.$$

If $L_g L_f h(x) \equiv 0$ for every x around x^*, we can go to the next higher-order derivative, and this way we can proceed further and get higher order ~~derivatives as th~~ integrators as the closed loop dynamics. However, if $L_g L_f^k h(x)$ is zero somewhere in a neighborhood and non-zero elsewhere things will be more complicated. The notion of strict relative degree provides a guide on how many derivatives of y one shall consider.

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→ Strict relative degree:—

The system (1) has a strict relative degree γ at $x_0 \in U$ (U is a neighborhood around x^*) if —

$$L_g L_f^i h(x) \equiv 0 \quad \forall\, x \in U \quad 0 \le i \le \gamma - 2$$

and, $L_g L_f^{\gamma-1} h(x_0) \neq 0$.

— Clearly $1 \le \gamma \le n$.

— The second condition implies that we can always find a neighborhood of x_0 such that $L_g L_f^{\gamma-1} h(x)$ is bounded away from zero over that neighborhood. This is a consequence of the assumption that f, g are smooth vector fields and h is a smooth function.