Lecture 22.

MAE 544
NONLINEAR CONTROL.

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Normal Form for SISO Systems :—

Consider the system —

$$\boxed{\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned}} \qquad \text{—— (1)}$$

where $x \in \mathbb{R}^n$, $f, g \in \mathfrak{X}(\mathbb{R}^n)$, $h \in C^\infty(\mathbb{R}^n)$ and $y, u \in \mathbb{R}^n$.
Assume (1) has a relative degree $r \leq n$ at $x_0 \in \mathbb{R}^n$ and $U \subset \mathbb{R}^n$ is an open neighborhood around x_0. Then we can find coordinate transformation such (1) can be expressed as—

NORMAL FORM

$$\boxed{\begin{aligned} \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= \xi_3 \\ &\vdots \\ \dot{\xi}_r &= b(\xi, \eta) + a(\xi, \eta)u \\ \dot{\eta} &= q(\xi, \eta) \\ \hline y &= \xi_1 \end{aligned}} \qquad \text{—— (2)}$$

where, $\eta \in \mathbb{R}^{n-r}$. In what follows we will see how we can obtain ξ and η from (1).

Let's first introduce some notations :—

$$ad_f^0 g = g \; ; \; ad_f^1 g = [f, g] \; ; \; ad_f^k g = [f, ad_f^{k-1} g] \;, \; k \geq 1$$

Also, thinking about vector fields as derivations over the space of smooth functions $C^\infty(\mathbb{R}^n)$ we have (as seen in Lecture 9-10) —

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$$L_{[f,g]}\lambda = L_f L_g \lambda - L_g L_f \lambda,$$

where, $f, g \in \mathcal{X}(\mathbb{R}^n)$ and $\lambda \in C^\infty(\mathbb{R}^n)$.

- $L_g L_f^k h(x) \equiv 0$, $0 \le k \le \mu$, $\forall\, x \in U$ if and only if $L_{ad_f^k g} h(x) \equiv 0$, $0 \le k \le \mu$ $\forall\, x \in U$.

⇒ Clearly it holds true if $\mu = 0$, as —

$$\text{RHS} \Leftrightarrow L_g h(x) \equiv 0 \Leftrightarrow \text{LHS}.$$

Note that,

$$L_{ad_f^k g} h(x) = L_{[f,\, ad_f^{k-1} g]} h(x)$$

$$= L_f L_{[ad_f^{k-1} g} h(x) - L_{ad_f^{k-1} g} L_f h(x)$$

Then for $k = 1$, we have —

$$L_{ad_f^1 g} h(x) = L_f L_g h(x) - L_g L_f h(x) \equiv 0 \quad \forall\, x \in U$$

$$\Updownarrow$$

$$L_g h(x) \equiv 0,\ L_g L_f h(x) \equiv 0 \quad \forall\, x \in U$$

For $k = 2$ we have —

$$L_{ad_f^2 g} h(x) = L_f L_{ad_f^1 g} h(x) - \left(L_f L_g - L_g L_f\right) L_f h(x)$$

$$= L_f L_{ad_f^1 g} h(x) - L_f L_g L_f h(x) + L_g L_f^2 h(x)$$

When, $L_{ad_f^1 g} h(x) \equiv 0$,

$$L_{ad_f^2 g} h(x) \equiv 0 \Longleftrightarrow L_g L_f h(x) \equiv 0 \text{ and } L_g L_f^2 h(x) \equiv 0$$

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The rest follows directly from recursion.

- Now we define,

$$\boxed{\begin{aligned} \phi_1(x) &= h(x) \\ \phi_2(x) &= L_f h(x) \\ &\vdots \\ \phi_r(x) &\equiv L_f^{r-1} h(x) \end{aligned}}$$

In absence of any input (i.e $u=0$), ϕ_i's can be interpreted as the output of (1) together with its first $r-1$ derivatives. Now, as r is the relative degree of (1), ϕ_i's represent y along with its first $r-1$ derivatives even when input is non-zero.

We have, $L_g L_f^j h(x) \equiv 0 \quad \forall x \in U \quad \cancel{\alpha} j \leq \cancel{r}-2$

Now,

$$L_{ad_f^i g} L_f^j h(x)$$

$$= L_f L_{ad_f^{i-1} g} L_f^j h(x) - L_{ad_f^{i-1} g} L_f L_f^j h(x)$$

$$= L_f L_{ad_f^{i-1} g} L_f^j h(x) - L_{ad_f^{i-1} g} L_f^{j+1} h(x)$$

$$= L_f \left(L_f L_{ad_f^{i-2} g} - L_{ad_f^{i-2} g} L_f \right) L_f^j h(x) - L_{ad_f^{i-1} g} L_f^{j+1} h(x)$$

$$= \underbrace{L_f^2 L_{ad_f^{i-2}g} L_f^j h(x) - L_f L_{ad_f^{i-2}g} L_f^{j+1} h(x) - L_{ad_f^{i-1}g} L_f^{j+1} h(x)}$$

This will eventually turned into terms of the form

{ As L_g can appear only once in the individual terms

$$L_f^{(\cdot)} L_g^{\boxed{(1)}} L_f^{(\cdot)} h(x) \quad \text{where} \quad l \leq i+j-1$$

Now assume $i+j \leq r-2$, then we can show that,

$$\boxed{L_{ad_f^i g} L_f^j h(x) = (-1)^i L_g L_f^{i+j} h(x)}$$

$$\equiv 0 \quad \forall x \in U$$

Moreover, when, $i+j = r-1$,

$$L_{ad_f^i g} L_f^j h(x_0) = (-1)^i L_g L_f^{i+j} h(x_0) \neq 0$$

From our definition of cotangent vector —

$$L_g h(x_0) = g(x_0)(h) = \underbrace{dh(x_0)}_{\in T^*_{x_0}\mathbb{R}^n}\underbrace{(g(x_0))}_{\in T_{x_0}\mathbb{R}^n}$$

$$\longrightarrow L_{ad_f^i g} L_f^j h(x_0) = dL_f^j h(x_0)\left(ad_f^i g(x_0)\right)$$

Thus we have —

$$\begin{bmatrix} dh(x_0) \\ dL_f h(x_0) \\ dL_f^2 h(x_0) \\ | \\ dL_f^{r-1} h(x_0) \end{bmatrix} \begin{bmatrix} g(x_0) & ad_f^1 g(x_0) & \cdots & ad_f^{r-1} g(x_0) \end{bmatrix} = \begin{bmatrix} 0 & & L_{ad_f^{r-1}g} L_f^0 h \\ & L_g L_f^{r-2} h(x_0) & \\ L_g L_f^{r-1} h(x_0) & & * \end{bmatrix}$$

Clearly the entries of this anti-diagonal are equal $\pm L_g L_f^{r-1} h(x_0) \neq 0$, and hence we can conclude that $d\phi_i(x_0)$'s are linearly independent at x_0 (which in turn implies linear independence in a neighborhood of x_0). In a similar way, "$ad_f^k g(x_0)$" $0 \leq k \leq r-1$ are linearly independent around x_0.

→ Codistribution and Frobenius Theorem:—

Codistribution (P) on a manifold (M) assigns a subspace of the cotangent space at every $p \in M$, i.e. $P(p) \subset T_p^* M$. P is smooth if it is locally spanned by smooth 1-forms, and P is non-singular if dimension of $P(p)$ is constant across the manifold. Also, $\Gamma(P)$ is the space of 1-forms that belong to the codistribution P, i.e. $\omega \in \Gamma(P)$ if $\omega(p) \in P(p)$ for every $p \in M$.

Given a smooth codistribution P, its kernel is defined as —

$$P^{\perp}(p) = \underbrace{\ker(P)}_{\text{Distribution}}(p) = \left\{ \text{span}\left\{ X(p) \,\middle|\, X \in \mathfrak{X}(M),\ \omega_p(X_p) = 0 \ \forall \omega \in \Gamma(P) \right\} \right\}$$

In a similar way, given a smooth distribution D, its annihilator is defined as —

$$D^{\perp}(p) = \underbrace{\text{ann}(D)}_{\text{Codistribution}}(p) = \text{span}\left\{ \tau(p) \,\middle|\, \tau \in \Omega^1(M),\ \tau_p(X_p) = 0 \ \forall X \in \Gamma(D) \right\}.$$

Moreover, if P (resp. D) is a non-singular codistribution (resp. distribution) then $\ker(P)$ (resp. $\text{ann}(D)$) is also non-singular. By definition, $D \subseteq \ker(\text{ann}(D))$ and $P \subseteq \text{ann}(\ker(P))$, and they are equal if they are non-singular (i.e. they have constant dimension). A codistribution is involutive if its kernel $\ker(P)$ is an involutive distribution.

Now, assume D is non-singular, smooth distribution which is involutive as well. Then using Frobenius theorem, we can show that, ~~around~~ any $p \in M$, there is a chart $(U, (x_1, \dots, x_m))$ such that —

$$D(q) = \text{span}\left\{ \frac{\partial}{\partial x_1}\Big|_q, \dots, \frac{\partial}{\partial x_d}\Big|_q \right\} \subset T_pM, \quad q \in U$$

where, d is the dim. of D.

Alternative version of Frobenius Theorem:—

A nonsingular distribution D (of dim. d) is integrable if and only if there exist $n-d$ real-valued functions $\lambda_1, \dots, \lambda_{n-d}$ on M such that —

$$D^\perp = \text{ann}(D) = \text{span}\{d\lambda_1, d\lambda_2, \dots, d\lambda_{n-d}\}.$$

→ Now, since D is non-singular, it can be represented as,

$$D(p) = \text{span}\{f_1(x), \dots, f_d(x)\}$$

where, $f_1, \dots, f_d \in \mathfrak{X}(M)$. Thus, $\lambda_1, \dots, \lambda_{n-d}$ can be obtained by solving the following PDEs in the local coordinate —

$$\frac{\partial \lambda_i}{\partial x}\left(f_1(x), \dots, f_d(x)\right) = 0, \quad 1 \le i \le n-d.$$

Back to Normal Form:—

As the system of our consideration (1) has relative degree $0 \le \gamma \le n$, we have —

$$g(x_0) \neq 0,$$

and hence, the distribution

$$\Delta = \text{span}\{g\}$$

is non-singular around x_0. Being 1-dimensional, it is involutive by defn. Then, from Frobenius Theorem (alternative version), we can conclude that there exist $(n-1)$ real valued functions $\eta_1(x), \eta_2(x), -, \eta_{n-1}(x)$ defined over a neighborhood around x_0 such that

$$\Delta^\perp = \text{Span}\{d\eta_1, ---, d\eta_{n-1}\}.$$

As $d\eta_i \in \Delta^\perp$,

$$0 = d\eta_i(g) = g(\eta_i) = L_g\eta_i \quad \forall x \text{ in a nbd. around } x_0.$$

- Claim: $\dim\left(\Delta^\perp + \text{span}\{dh, dL_fh, --, dL_f^{\gamma-1}h\}\right)(x_0) = n$

Proof: We can prove this by contradiction. Suppose the claim is not true. Then, there exists —

$$g(x_0) \in \left(\text{span}\{dh, dL_fh, --, dL_f^{\gamma-1}h\}\right)^\perp(x_0) \neq \text{non-empty}$$

This implies,

$$g(x_0)\left(dL_f^k h(x_0)\right) = 0 \quad 0 \le k \le \gamma-1$$

$$\Leftrightarrow L_g L_f^k h(x_0) = 0 \quad 0 \le k \le \gamma-1$$

But γ is the relative degree,

$$L_g L_f^{\gamma-1} h(x_0) \neq 0.$$

Thus it is a contradiction, and that proves our claim.

Now, from this pool of $(n-1)$ function we choose $n-\gamma$ of them, and call them $\eta_1, \eta_2, -, \eta_{n-\gamma}$ (without loss of generality. such that —

$$\begin{pmatrix} dh \\ | \\ dL_f^{\gamma-1}h \\ d\eta_1 \\ | \\ d\eta_{n-\gamma} \end{pmatrix}$$

are linearly independent at x_0. Also that value of η_i at x_0 can be chosen arbitrarily (because any $\tilde{\eta}_i$ defined as $\tilde{\eta}_i(x) = \eta_i(x) + c_i$, c_i — constant, will satisfy the same properties).

Then can define a coordinate transformation as —

$$\Phi : x \longmapsto \begin{pmatrix} \xi_1 \\ \xi_2 \\ \\ \xi_\gamma \\ \eta_1 \\ | \\ \eta_{n-\gamma} \end{pmatrix} \triangleq \begin{pmatrix} h(x) \\ L_f h(x) \\ | \\ L_f^{\gamma-1} h(x) \\ \eta_1(x) \\ | \\ \eta_{n-\gamma}(x) \end{pmatrix}$$

It is easy to check that —

$$\dot{\xi}_1 = L_f h(x) = \xi_2$$

$$|$$

$$\dot{\xi}_\gamma = L_f L_f^{\gamma-1} h(x) + u L_g L_f^{\gamma-1} h(x) = \underbrace{L_f^{\gamma} h(\Phi^{-1}(\xi,\eta))}_{b(\xi,\eta)} + u \underbrace{L_g L_f^{\gamma-1} h(\Phi^{-1}(\xi,\eta))}_{a(\xi,\eta)}$$

$$\dot{\eta}_i = L_f \eta_i(x) = L_f \eta_i(\Phi^{-1}(\xi,\eta)) \quad (\text{as } L_g \eta_i = 0)$$

Thus we have transformed (1) into (2).

An Example:—

$$\dot{x} = \underbrace{\begin{pmatrix} x_1^3 \\ \cos x_1 \cos x_2 \\ x_2 \end{pmatrix}}_{f(x)} + \underbrace{\begin{pmatrix} \cos x_2 \\ 1 \\ 0 \end{pmatrix}}_{g(x)} u$$

$$y = h(x) = x_3$$

Clearly,

$$L_g h(x) = 0; \quad L_g L_f h(x) = 1$$

$$L_f h(x) = x_2; \quad L_f^2 h(x) = \cos x_1 \cos x_2$$

Thus, this system has relative degree = 2, and hence we have,

$$\xi_1 = h(x) = x_3 \; ; \; \xi_2 = L_f h(x) = x_2$$

Also, we can solve η from —

$$\begin{bmatrix} \dfrac{\partial \eta}{\partial x_1} \\ \dfrac{\partial \eta}{\partial x_2} \end{bmatrix}^T \begin{bmatrix} \cos x_2 \\ 1 \end{bmatrix} = 0 \Leftarrow \begin{matrix} \dfrac{\partial \eta}{\partial x_1} = 1 \\ \dfrac{\partial \eta}{\partial x_2} = \cos x_2 \end{matrix} \Leftarrow \eta = x_1 - \sin x_2$$

$$\dot{\eta} = \dot{x}_1 - \cos x_2 \, \dot{x}_2 = x_1^3 - \cos^2 x_2 \cos x_1$$

$$= (\eta + \sin \xi_2)^3 - \cos^2 \xi_2 \cos(\eta + \sin \xi_2)$$

Thus we have the following 1-form —

$$\dot{\xi}_1 = \xi_2$$

$$\dot{\xi}_2 = \cos(\eta + \sin \xi_2) \cos \xi_2 + u$$

$$\dot{\eta} = (\eta + \sin \xi_2)^3 - \cos^2 \xi_2 \cos(\eta + \sin \xi_2)$$