Lecture 2

MAE 544

NONLINEAR CONTROL.

12/6/2017
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Exact State Space Linearization:

Consider the system —

$$\dot{x} = f(x) + u\,g(x)$$
$$y = h(x)$$

where, $x \in \mathbb{R}^n$; $y, u \in \mathbb{R}$; $f, g \in \mathfrak{X}(\mathbb{R}^n)$; $h \in C^\infty(\mathbb{R}^n)$. Assume that the system has relative degree $\gamma = n$. Then its normal form representation will look like —

$$\dot{\xi}_1 = \xi_2$$
$$\dot{\xi}_2 = \xi_3$$
$$\vdots$$
$$\dot{\xi}_n = b(\xi) + u\,a(\xi)$$

and, $y = \xi_1$

where, $a(\xi) = L_g L_f^{n-1} h(x)$ and $b(\xi) = L_f^n h(x)$.

Then by choosing u as —

$$u = \left(\frac{1}{a(\xi)}\right)(v - b(\xi)),$$

we have the following linear dynamics —

$$\dot{\xi} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \xi + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} v$$

$$y = (1, 0, 0, \cdots, 0)\,\xi$$

Then one can design a state feedback controller —

$$v = \sum_{i=1}^{n} \alpha_i \xi_i = \sum_{i=1}^{n} \alpha_i L_f^{i-1} h(x),$$

and place the closed-loop poles arbitrarily.

12/6/2017
BD | 23-2

→ Now we consider a system for which the output (i.e the function h) is yet to be defined. Can we find an $h \in C^{\infty}(\mathbb{R}^n)$ such the resulting relative degree is precisely n?

The key idea for state space exact linearization is follows —

Given a system $\boxed{\dot{x} = f(x) + u g(x)}$ and a point $x_0 \in \mathbb{R}^n$, find a feedback $\boxed{u = \alpha(x) + v\beta(x)}$ defined over a neighborhood U around x_0 and a coordinate transformation $\boxed{z = \Phi(x)}$ defined on U, such that the closed-loop dynamics is linear and controllable, i.e

$$\dot{z} = Az + Bv$$

where,
$$A = \frac{\partial \Phi}{\partial x}\Big(f(x) + g(x)\,\alpha(x)\Big)\Big|_{x = \Phi^{-1}(z)}$$

and,
$$B = \frac{\partial \Phi}{\partial x}\Big(g(x)\beta(x)\Big)\Big|_{x = \Phi^{-1}(z)}$$

→ Let, $z = \Phi(x)$ be a coordinate transformation for the system — $\dot{x} = f(x) + u g(x)\,;\; y = h(x)$

Then, $\dot{z} = \bar{f}(z) + u\,\bar{g}(z)$ where, $\bar{f}(z) = \left(\frac{\partial \Phi}{\partial x} f(x)\right)\Big|_{x = \Phi^{-1}(z)}$

and, $\bar{g}(z) = \left(\frac{\partial \Phi}{\partial x} g(x)\right)\Big|_{x = \Phi^{-1}(z)}$

~~Hence,~~

$\cancel{L_{\bar{f}}\bar{h}(z) = \frac{\partial \bar{h}}{\partial z}\bar{f}(z)}$

Also, $\bar{h}(z) = h\left(\Phi^{-1}(z)\right)$

Hence,

$$L_{\bar{f}} \bar{h}(z) = \frac{\partial \bar{h}}{\partial z} \bar{f}(z)$$

$$= \left.\frac{\partial h}{\partial x}\right|_{x=\Phi^{-1}(z)} \left(\frac{\partial \Phi^{-1}}{\partial z}\right) \left(\frac{\partial \Phi}{\partial x} f(x)\right)\Big|_{x=\Phi^{-1}(z)}$$

$$= \left.\frac{\partial h}{\partial x} f(x)\right|_{x=\Phi^{-1}(z)} = L_f h(x)\big|_{x=\Phi^{-1}(z)}.$$

In a similar way, $L_{\bar{g}} L_{\bar{f}}^k \bar{h}(z) = \left(L_g L_f^k h(x)\right)_{x=\Phi^{-1}(z)}$

This implies that the relative degree of a system does not change under coordinate transformation.

→ The closed loop dynamics will be—

$$\dot{x} = [f(x) + \alpha(x) g(x)] + v[\beta(x) g(x)]$$

under the feedback—

$$u = \alpha(x) + v\beta(x).$$

Clearly, $L^0_{f+\alpha g} h(x) = L^0_f h(x)$.

Now, assume, $L^k_{f+\alpha g} h(x) = L^k_f h(x)$ holds for some $0 \le k < \gamma - 1$. Then,

$$L^{k+1}_{f+\alpha g} h(x) = L_{f+\alpha g} L^k_{f+\alpha g} h(x)$$

$$= L_{f+\alpha g} L_f^k h(x) = L_f L_f^k h(x) + L_{\alpha g} L_f^k h(x)$$

$$= L_f^{k+1} h(x) + \alpha(x) \cdot \underbrace{L_g L_f^k h(x)}_{=0}$$

Thus, $L^{k+1}_{f+\alpha g} h(x) = L_f^{k+1} h(x)$

This implies that relative degree is invariant under feedback.

12/6/2017
BD | 23-4

→ Earlier we have shown (by construction) that ~~that~~ state-space exact linearization is possible if the relative degree $\gamma = n$. This condition is necessary as well (which can be shown by exploiting the invariance of γ under feedback and coordinate transformation).

But how we can find an appropriate output $h \in C^\infty(\mathbb{R}^n)$ such that the relative degree is n, i.e. —

$$L_g h(x) = L_g L_f h(x) = \cdots = L_g L_f^{n-2} h(x) = 0 \quad \forall x \in U$$

and, $L_g L_f^{n-1} h(x_0) \neq 0$

Alternatively, by using $ad_f^k : \mathfrak{X}(\mathbb{R}^n) \to \mathfrak{X}(\mathbb{R}^n)$ notation, we can express these conditions as —

$$\boxed{L_g h(x) = L_{ad_f g} h(x) = \cdots = L_{ad_f^{n-2} g} h(x) = 0 \quad \forall x \in U} \quad (1)$$

and, $\boxed{L_{ad_f^{n-1} g} h(x_0) \neq 0}$ (2)

Lemma: There exists an $h \in C^\infty(\mathbb{R}^n)$ defined over U ~~s~~ satisfying (1)-(2) if and only if —

(i) $\{ad_f^0 g(x_0), ad_f^1 g(x_0), \text{—}, ad_f^{n-1} g(x_0)\}$ are linearly independent, and

(ii) The distribution $D \triangleq \mathrm{span}\{ad_f^0 g, ad_f^1 g, \text{—}, ad_f^{n-2} g\}$ is involutive.

Proof:

$\Leftarrow$ | **If-part:**

Suppose (i) holds. Then D is nonsingular with dimension $(n-1)$ around x_0.
If (ii) holds in addition, then there exists a function $h \in C^\infty(\mathbb{R}^n)$ such that dh spans $D^\perp$ in a neighborhood around x_0, i.e. (1) holds true.
Then (2) also must hold true. Otherwise, we have —

$$dh(x_0)\left[g(x_0) \quad ad_f g(x_0) \text{——} ad_f^{n-1} g(x_0)\right] = 0$$

for a non-trivial h, which is a contradiction.

$\Rightarrow$ | **Only-if Part:**

Suppose there exists a function h, such that (1)-(2) hold. Then, from Last lecture (discussion on the anti lower triangular matrix), we know that $\{g, ad_f g, ---, ad_f^{n-1} g\}$ are linearly independent around x_0. This means (i) holds and D is non-singular around x_0 with dimension $(n-1)$.
As (1) equivalent to —

$$dh(x)\left[g(x) \quad ad_f g(x) \; --- \; ad_f^{n-2} g(x)\right] = 0 \; \forall x \in U,$$

$dh(x)$ spans $D^\perp$ around x_0. Then, from Frobenius we can conclude that D is involutive around x_0.

Thus we can conclude that (i) and (ii) provide a necessary + sufficient cond^n for solving the state-space exact linearization problem.

To wrap-up, following is a step-by-step recipe for state-space exact linearization — 12/6/2017 BD/(23-6)

— From f and g compute $ad_f^k g$, $0 \le k \le n-1$, and verify (i) & (ii).

— If they hold true solve

$$\frac{\partial h}{\partial x}(x)\left(ad_f^k g(x)\right) = 0 \qquad 0 \le k \le n-2$$

to obtain the appropriate output function h.

— Set:

$$u = \frac{1}{L_g L_f^{n-1} h(x)}\left[v - L_f^{n} h(x)\right] \quad \forall x \in U$$

— Set linearizing coordinate transformation:

$$\Phi(x) = \left(h(x), L_f h(x), \text{—}, L_f^{n-1} h(x)\right)$$

- It is easy to check that controllabity of the pair $\left(\frac{\partial f}{\partial x}(x_0), g(x_0)\right)$ is equivalent (i). On the other hand, for a planar system (i.e. $n=2$), (ii) always holds true. Therefore, exact feedback linearization is possible if and only if $\left(\frac{\partial f}{\partial x}(x_0), g(x_0)\right)$ is controllable.

Zero-Dynamics :—

The exact input-output linearization control law for a system with relative degree γ, makes the closed loop dynamics behave like a γ-order integrator. As a consequence, we have $(n-\gamma)$ states unobservable.

In a linear system context this corresponds

12/6/2017
BD/23-7

to shifting $(n-\gamma)$-poles of the closed-loop system $(n-\gamma)$-zeros of the open-loop dynamics and moving the rest γ-poles to the origin. Thus one can perceive the input-output linearizing control law as a non-linear counterpart of zero-cancelling control law.

→ **Output-Zeroing Problem:-**

Find (if possible) an initial state $\bar{x} \in U$ and an input $\bar{u}(t)$, $t \geq 0$ such that $y(t) \equiv 0 \;\forall t \geq 0$.

$y(t) \equiv 0 \Rightarrow \xi_1 \equiv \xi_2 \equiv \cdots \equiv \xi_\gamma \equiv 0$ (From normal form)

This means,

$$\dot{\xi}_\gamma = 0 = b(0,\eta) + u\,a(0,\eta) \Rightarrow u = -\frac{b(0,\eta)}{a(0,\eta)}$$

where, $\eta(t)$ is a solution of —

$$\dot{\eta} = q(0,\eta)$$

Thus we can conclude that to keep the output held at zero, we need —

$$\bar{x} \in M \triangleq \left\{ x \in \mathbb{R}^n \,\middle|\, L_f^k h(x) = 0 \;\forall\, 0 \leq k \leq \gamma-1 \right\}$$

and,

$$\bar{u}(x) = -\frac{L_f^\gamma h(x)}{L_g L_f^{\gamma-1} h(x)}.$$

Clearly M is invariant under this feedback. The restriction of the dynamics on M is called zero-dynamics.