A brief on controllability of nonlinear systems

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Abstract

Results concerning the controllability of control affine systems are reviewed. The discussion starts with accessibility results connected with Lie algebraic ideas, and winds its way to some recent local controllability results.

1. Introduction

One of the very basic questions in control system theory is, “Where can I go from here?” This question has a nice answer in some simple cases, but the general case remains open. It is our intention to make clear the question, and provide some answers, most of them well-known.

Let us define the systems we look at. We consider systems of the type

$$\dot{x}(t) = f_0(x(t)) + \sum_{a=1}^{m} u^a(t) f_a(x(t)) \tag{1.1}$$

where $t \mapsto x(t)$ is a curve in a state manifold M (no harm will arise in thinking of M as being an open subset of $\mathbb{R}^n$, as our treatment is local). The vector field f_0 is the ***drift vector field***, describing the dynamics of the system in the absence of controls, and the vector fields $f_1, \dots, f_m$ are the ***input vector fields*** or ***control vector fields***, indicating how we are able to actuate the system. The vector fields $f_0, f_1, \dots, f_m$ are assumed to be real analytic, although some of the results hold for C^∞ vector fields. We will try to point out the distinctions when they arise. We do not ask for any sort of linear independence of the control vector fields $f_1, \dots, f_m$. We shall suppose that the controls $u\colon [0, T] \to U$ are locally integrable with U some subset of $\mathbb{R}^m$. Common examples are

$$U = \mathbb{R}^m, \quad U = \{u \in \mathbb{R}^m \mid \|u\| = 1\}, \quad U = [-1, 1]^m.$$

We shall have some things to say about the nature of the control set U as we go along. We allow the length T of the interval on which the control is defined to be arbitrary. It will be convenient to denote by $\tau(u)$ the right endpoint of the interval for a given control u. For a fixed U we denote by $\mathscr{U}$ the collection of all measurable controls taking their values in U. To be concise about this, a ***control affine system*** is a triple $\Sigma = (M, \mathscr{F} = \{f_0, f_1, \dots, f_m\}, U)$ with all objects as defined above. A ***controlled trajectory*** for Σ is a pair (c, u) where $u \in \mathscr{U}$ and where $c\colon [0, \tau(u)] \to M$ is defined so that

$$c'(t) = f_0(c(t)) + \sum_{a=1}^{m} u^a(t) f_a(c(t)).$$

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One can show that for admissible controls, the curve c will exist at least for sufficiently small times, and that the initial condition $c(0) = x_0$ uniquely defines c on its domain of definition.

For $x \in M$ and $T > 0$ we define

$$\mathcal{R}_\Sigma(x, T) = \{c(T) \mid (c, u) \text{ is a controlled trajectory for } \Sigma \text{ with } \tau(u) = T \text{ and } c(0) = x\}$$

and

$$\mathcal{R}_\Sigma(x, \leq T) = \bigcup_{t\in[0,T]} \mathcal{R}_\Sigma(t, x), \quad \mathcal{R}_\Sigma(x) = \bigcup_{t\geq 0} \mathcal{R}_\Sigma(x, t).$$

These are each various types of ***reachable sets***. With these in hand, we can provide definitions for various types of controllability.

1.1 Definition: Let $\Sigma = (M, \mathscr{F}, U)$ be a control affine system and let $x \in M$.

(i) Σ is ***accessible*** from x if $\mathrm{int}(\mathcal{R}_\Sigma(x)) \neq \emptyset$.

(ii) Σ is ***strongly accessible*** from x if $\mathrm{int}(\mathcal{R}_\Sigma(x, T)) \neq \emptyset$ for each $T > 0$.

(iii) Σ is ***locally controllable*** from x if $x \in \mathrm{int}(\mathcal{R}_\Sigma(x))$.

(iv) Σ is ***small-time locally controllable*** (***STLC***) from x if there exists $T > 0$ so that $x \in \mathrm{int}(\mathcal{R}_\Sigma(x, \leq t))$ for each $t \in]0, T]$.

(v) Σ is ***globally controllable*** from x if $\mathcal{R}_\Sigma(x) = M$. •

There are actually almost as many notions of controllability as there are people who do research in the field. However, the notions of accessibility are, as we shall see, pretty well nailed down. When talking about controllability, one needs to be clear about what controllability means. This is in contrast to linear systems where, at least if one allows unbounded controls, many notions of controllability are equivalent to the Kalman rank condition.

Let us look at a few examples which distinguish at least some of the controllability definitions we give.

1.2 Examples: 1. Here is the typical simple example of a system that is accessible but not controllable. We take $M = \mathbb{R}^2$, $m = 1$, $U = [-1, 1]$, and the control system

$$\begin{aligned}\dot{x} &= u\\ \dot{y} &= x^2.\end{aligned}$$

This system is (not obviously) accessible from $(0, 0)$, but is (obviously) not locally controllable from that same point. The character of the reachable sets is shown in Figure 1.[1] Note that although $\mathcal{R}_\Sigma((0, 0), \leq T)$ has nonempty interior, the initial point $(0, 0)$ is not in that interior. Thus this is a system that is not controllable in any sense. Note that the system is also strongly accessible.

[1]It might be an interesting exercise to show that the left and right boundaries for $\mathcal{R}_\Sigma(x)$ are given by the graph of the function $y(x) = \frac{1}{3}|x|^3$ and that the upper boundary for $\mathcal{R}_\Sigma((0, 0), T)$ is given by the graph of the function

$$y(x) = -\frac{|x|^3}{4} + \frac{T|x|^2}{4} + \frac{T^2|x|}{4} + \frac{T^3}{12}.$$

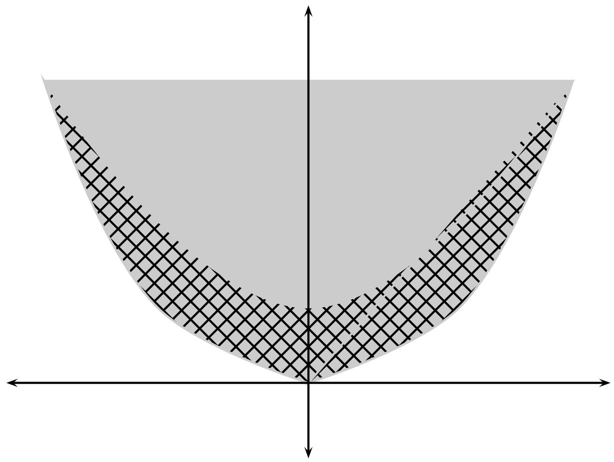

Figure 1. Reachable sets: the shaded area represents $\mathcal{R}_\Sigma((0,0))$ and the hatched area represents $\mathcal{R}_\Sigma((0,0),T)$ for some $T>0$

2. Let us now look at an example that is accessible but not strongly accessible. We take $M=\mathbb{R}^2$, $m=1$, $U=\mathbb{R}$, and consider the control system

$$\begin{aligned}\dot{x} &= u\\ \dot{y} &= 1.\end{aligned}$$

In Figure 2 we show the reachable sets. Note that the system is (fairly obviously)

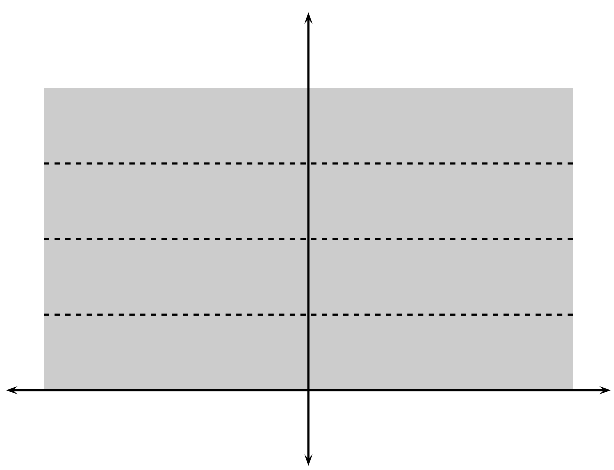

Figure 2. Reachable sets: the shaded region represents $\mathcal{R}_\Sigma((0,0))$ and the dashed lines represent $\mathcal{R}_\Sigma((0,0),T)$ for various T's

accessible, but (obviously) not strongly accessible. The system is also not controllable in any of the three senses we define.

3. Next we consider a system that is locally controllable, but not STLC. We take $M=$

$\mathbb{R}\times\mathbb{S}^1$ with coordinates (x,θ), $m=1$, $U=[-1,1]$, and defined by the control system

$$\dot{x} = u$$
$$\dot{\theta} = 1.$$

The reachable sets are shown in Figure 3, and we can see there that for small times the

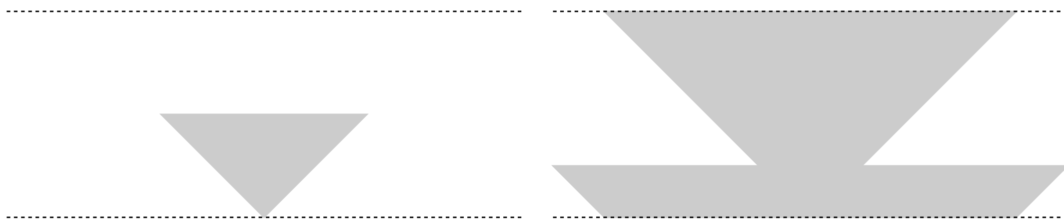

Figure 3. Reachable set in small time (left) and larger time (right)

reachable set from $(0,0)$ does not contain $(0,0)$ in its interior, but that the reachable set for large times does contain $(0,0)$ in its interior. Indeed, one can readily see that this system is globally controllable, although not STLC. •

2. Accessibility theory

Let us first turn our attention to determining when a system Σ is accessible. An essential part of this discussion is the Lie algebraic properties of vector fields. Thus we begin with these.

2.1. Orbits of families of vector fields. We denote by $\Gamma(TM)$ the collection of analytic vector fields on M. Thus $X\in\Gamma(TM)$ is a real analytic map $X\colon M\to TM$ having the property that $X(x)\in T_xM$. We let $\mathscr{V}\subset\Gamma(TM)$ be an arbitrary family of vector fields. Given a control affine system $\Sigma=(M,\mathscr{F},U)$ there is an associated family of vector fields

$$\mathscr{V}_\Sigma = \Big\{ f_0 + \sum_{a=1}^m u^a f_a \;\Big|\; u\in U\Big\}.$$

Recall that an ***integral curve*** for a vector field X is a curve $c\colon[0,T]\to M$ for which $c'(t)=X(c(t))$ for each $t\in[0,T]$. We define the ***flow*** of X to be the map $\exp_X\colon\mathbb{R}\times M\to M$[2] given by $\exp_X(t,x)=c(t)$ where c is the integral curve for X satisfying $c(0)=x$. It is convenient notation to write $\exp_X(t,x)=e^{tX}(x)$. For a family $\mathscr{V}$ of vector fields, we denote by $\mathrm{Diff}(\mathscr{V})$ the subgroup of the diffeomorphism group of M generated by elements of the form

$$e^{t_1X_1}\circ\cdots\circ e^{t_kX_k}(x),\quad t_1,\dots,t_k\in\mathbb{R},\ X_1,\dots,X_k\in\mathscr{V},\ k\in\mathbb{N}.$$

Thus, a generator of this form applied to x sends x to the point obtained by flowing along X_k for time t_k, then along X_{k-1} for time t_{k-1}, and so on down to flowing along X_1 for time t_1. The ***$\mathscr{V}$-orbit*** through x is the set

$$\mathcal{O}(x,\mathscr{V}) = \{\phi(x) \mid \phi\in\mathrm{Diff}(\mathscr{V})\}.$$

[2]We assume all vector fields to be complete so that there flows are defined on all of $\mathbb{R}$.

One can also do this for fixed times. We do this as follows. Define $\mathrm{Diff}_0(\mathscr{V})$ as the subgroup of the diffeomorphism group of M generated by elements of the form

$$e^{t_1 X_1} \circ \cdots \circ e^{t_k X_k}(x), \quad t_1, \dots, t_k \in \mathbb{R},\ \sum_{\alpha=1}^{k} t_\alpha = 0,\ X_1, \dots, X_k \in \mathscr{V},\ k \in \mathbb{N}.$$

This is a normal subgroup of $\mathrm{Diff}(\mathscr{V})$.[3] Now we let $X \in \mathscr{V}$ and define

$$\mathrm{Diff}_T(\mathscr{V}) = \{\phi \circ e^{TX} \mid \phi \in \mathrm{Diff}_0(\mathscr{V})\}.$$

Thus $\mathrm{Diff}_T(\mathscr{V})$ is the coset of $\mathrm{Diff}_0(\mathscr{V})$ through e^{TX}.[4] One may verify that this only depends on T and not on the choice of $X \in \mathscr{V}$. Indeed, one may verify that $\mathrm{Diff}_T(\mathscr{V})$ is simply that collection of diffeomorphisms in $\mathrm{Diff}(\mathscr{V})$ of the form

$$e^{t_1 X_1} \circ \cdots \circ e^{t_k X_k}(x), \quad t_1, \dots, t_k \in \mathbb{R},\ \sum_{\alpha=1}^{k} t_\alpha = T,\ X_1, \dots, X_k \in \mathscr{V},\ k \in \mathbb{N}.$$

However, our characterisation in terms of normal subgroups is helpful when we come to discuss what is essentially the Lie algebra for $\mathrm{Diff}_0(\mathscr{V})$. In any event, $\mathrm{Diff}_T(\mathscr{V})$ gives rise to the ***($\mathscr{V}, T$)-orbit*** through x:

$$\mathcal{O}_T(x, \mathscr{V}) = \{\phi(x) \mid \phi \in \mathrm{Diff}_T(\mathscr{V})\}.$$

This is all well and good. However, in control theory, time usually only goes forward. With this in mind we let $\mathrm{Diff}^+(\mathscr{V})$ be the semi-group of diffeomorphisms generated by elements of the form

$$e^{t_1 X_1} \circ \cdots \circ e^{t_k X_k}(x), \quad t_1, \dots, t_k \geq 0,\ X_1, \dots, X_k \in \mathscr{V},\ k \in \mathbb{N}.$$

For good measure, for $T \geq 0$ we also define $\mathrm{Diff}_T^+(\mathscr{V})$ as the semi-group generated by those elements of the form

$$e^{t_1 X_1} \circ \cdots \circ e^{t_k X_k}(x), \quad t_1, \dots, t_k \geq 0,\ \sum_{\alpha=1}^{k} t_\alpha = T,\ X_1, \dots, X_k \in \mathscr{V},\ k \in \mathbb{N}.$$

These semi-groups define subsets of $\mathcal{O}(x, \mathscr{V})$ given by

$$\mathcal{O}^+(x, \mathscr{V}) = \{\phi(x) \mid \phi \in \mathrm{Diff}^+(\mathscr{V})\}, \quad \mathcal{O}_T^+(x, \mathscr{V}) = \{\phi(x) \mid \phi \in \mathrm{Diff}_T^+(\mathscr{V})\}.$$

A family $\mathscr{V}$ of vector fields is ***attainable*** from x if $\mathrm{int}(\mathcal{O}^+(x, \mathscr{V})) \neq \emptyset$ and is ***strongly attainable*** if $\mathrm{int}(\mathcal{O}_T^+(x, \mathscr{V})) \neq \emptyset$ for each $T > 0$. These definitions obviously closely mirror the definitions of accessibility and strong accessibility.

Let us first describe the orbits $\mathcal{O}(x, \mathscr{V})$. This description is provided in varying degrees of generality by many authors [Hermann 1960, Hermann 1962, Matsuda 1968, Nagano 1966, Stefan 1974a, Stefan 1974b, Sussmann 1973]. The description hinges on the notion of the

[3] A subgroup H of a group G is ***normal*** when $ghg^{-1} \in H$ for each $g \in G$ and $h \in H$. With this definition, it is rather obvious that $\mathrm{Diff}_0(\mathscr{V})$ is a normal subgroup of $\mathrm{Diff}(\mathscr{V})$.

[4] If $H \subset G$ is a subgroup of a group G, the ***coset*** of H through $g \in G$ is the set $\{gh \mid h \in H\}$.

Lie bracket. We let $X, Y \in \Gamma(TM)$ and choose a local set of coordinates $(x^1, \dots, x^n)$ for M. The local forms for X and Y are then just vector functions of x. The ***Lie bracket*** $[X, Y]$ of X and Y is described by the vector function

$$[X, Y](x) = \boldsymbol{D}Y(x) \cdot X(x) - \boldsymbol{D}X(x) \cdot Y(x). \tag{2.1}$$

One may verify that this definition does not depend on the choice of coordinates, and so defines a vector field $[X, Y]$ on M. A good way to imagine this vector field is as follows. Construct a curve $c\colon [0, T] \to M$ as follows. Start at $x \in M$ and follow the integral curve of X for time $\frac{T}{4}$. Now follow the integral curve for Y for time $\frac{T}{4}$. Now follow the integral curve for $-X$ for time $\frac{T}{4}$. Finally, follow the integral curve for $-Y$ for time $\frac{T}{4}$. After doing this, you will end up at a point $c(T)$. If one does a Taylor expansion for $c(T)$ one finds that

$$c(T) = x + T^2[X, Y](x) + \text{h.o.t.s.}$$

Thus the Lie bracket measures the lowest-order effect of moving away from x using a trajectory of the type described. One readily verifies that the Lie bracket has the following properties:

1. the map $(X, Y) \mapsto [X, Y]$ is $\mathbb{R}$-bilinear;
2. $[Y, X] = -[X, Y]$;
3. $[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0$;
4. $[fX, Y] = f[X, Y] - (\mathscr{L}_X f)Y$ for a function f.

The third property is the ***Jacobi identity*** and is the only non-obvious property, although it is very important. On a $\mathbb{R}$-vector space, any product having the first three properties defines a Lie bracket on this vector space, and makes the vector space into a ***Lie algebra***. We know a lot about Lie algebras [Jacobson 1962, Serre 1992, Varadarajan 1974]. For a family of vector fields, let us denote by $\mathscr{L}(\mathscr{V})$ the smallest Lie subalgebra of $\Gamma(TM)$ that contains $\mathscr{V}$. If $\mathscr{V}$ is finite, say $\mathscr{V} = \{X_1, \dots, X_k\}$, then it turns out that all vector fields in $\mathscr{L}(\mathscr{V})$ are $\mathbb{R}$-linear combinations of vector fields of the form

$$[X_{i_1}, [X_{i_2}, \cdots, [X_{i_{\ell-1}}, X_{i_\ell}]]], \quad i_1, \dots, i_\ell \in \{1, \dots, k\}.$$

For $x \in M$ we then define

$$L(\mathscr{V})_x = \{X(x) \mid X \in \mathscr{L}(\mathscr{V})\}.$$

Thus $L(\mathscr{V})_x$ is a subspace of T_xM, and so $L(\mathscr{V})$ defines a distribution on M, in an appropriately general sense of the word "distribution." An ***integral manifold*** for $L(\mathscr{V})$ is a submanifold $N \subset M$ for which $T_xN \subset L_x(\mathscr{V})$ for each $x \in N$. An integral manifold containing $x \in M$ is the ***maximal integral manifold*** through x if it is a superset of any other integral manifold containing x. Because of the way $L(\mathscr{V})$ is constructed, there is no *a priori* reason to expect that $L(\mathscr{V})$ admits *any* integral manifolds, never mind allows a satisfactory *maximal* integral manifold. However, the miracle is that maximal integral manifolds are "nice," and that furthermore, they are the same as the orbits for $\mathscr{V}$. This is the content of the following result whose difficult proof can be gotten from the paper of Sussmann [1973].

2.1 Theorem: *If $\mathscr{V}$ is a family of complete analytic vector fields on M and $x \in M$, then the following statements are true:*

(i) $\mathcal{O}(x, \mathscr{V})$ is an analytic immersed[5] *submanifold;*

(ii) for each $y \in \mathcal{O}(x, \mathscr{V})$, $T_y(\mathcal{O}(x, \mathscr{V})) = L(\mathscr{V})_y$;

(iii) M is the disjoint union of all orbits of $\mathscr{V}$.[6]

This is one of those theorems which falls under the category of "important." If the vector fields in $\mathscr{V}$ are only C^∞, then one can generally only infer that for each $y \in \mathcal{O}(x, \mathscr{V})$, $L(\mathscr{V})_y \subset T_y(\mathcal{O}(x, \mathscr{V}))$.

Let

$$\dim(\mathscr{V}) = \max_{x \in M} \dim(\mathcal{O}(x, \mathscr{V})).$$

Generally speaking, $\dim(\mathcal{O}(x, \mathscr{V})) < \dim(\mathscr{V})$, and so it becomes interesting to know the set of points $x \in M$ for which $\dim(\mathcal{O}(x, \mathscr{V})) = \dim(\mathscr{V})$.

2.2 Theorem: *If M is connected, the set*

$$\{x \in M \mid \ \dim(\mathcal{O}(x, \mathscr{V})) = \dim(\mathscr{V})\}$$

is an open dense subset of M.

This says that the set of points where the integral manifolds have less that the maximum possible dimension is a "thin" subset. If the vector fields are only C^∞, then "open and dense" gets replaced with just "open."

Let us consider an example.

2.3 Example: We again take $M = \mathbb{R}^2$ and we define $\mathscr{V} = \{X_1, X_2\}$ with

$$X_1 = \begin{bmatrix} 0 \\ y \end{bmatrix}, \quad X_2 = \begin{bmatrix} x \\ 0 \end{bmatrix}.$$

This is a simple case since one verifies that $\mathscr{L}(\mathscr{V}) = \mathscr{V}$. There are nine integral manifolds:

1. $\{(0, 0)\}$;
2. $\{(x, 0) \mid \ x > 0\}$;
3. $\{(x, 0) \mid \ x < 0\}$;
4. $\{(0, y) \mid \ y > 0\}$;
5. $\{(0, y) \mid \ y < 0\}$;
6. $\{(x, y) \mid \ x > 0, \ y > 0\}$;
7. $\{(x, y) \mid \ x > 0, \ y < 0\}$;
8. $\{(x, y) \mid \ x < 0, \ y > 0\}$;
9. $\{(x, y) \mid \ x < 0, \ y < 0\}$.

Also see Figure 4. Note that we have integral manifolds of zero, one, and two-dimensions, and that the union of the integral manifolds of dimension two is indeed open and dense. •

Now let us turn to attainability and strong attainability. The results here are from the landmark paper of Sussmann and Jurdjevic [1972]. First let us consider the attainability result. First note that by Theorem 2.1 it is evident that if $\mathscr{V}$ is attainable from x then $L(\mathscr{V})_x = T_x M$. This condition is sufficient as well.

[5] An ***immersed submanifold*** of M is a subset $S \subset M$ for which there exists a manifold N, and an injective mapping $\iota\colon N \to M$ for which $S = \iota(N)$ and for which the derivative $T_y\iota$ has full rank for each $y \in N$.

[6] In other words, the collection of all orbits defines a foliation of M.

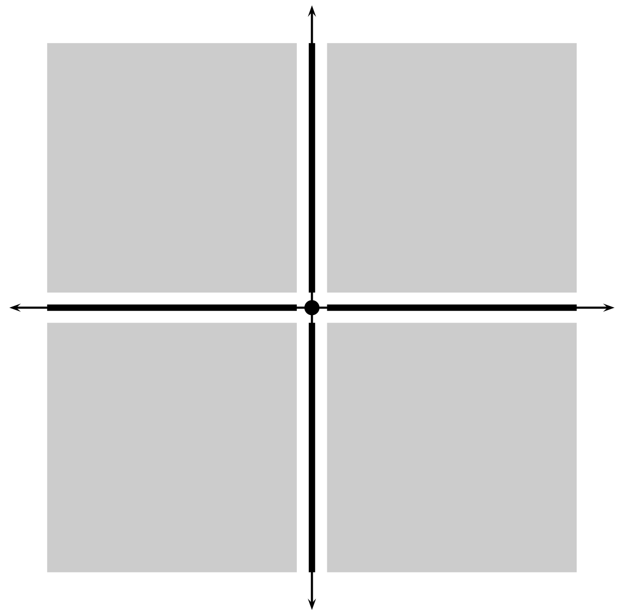

Figure 4. Integral manifolds

2.4 Theorem: *Let $\mathscr{V}$ be an analytic family of vector fields on M, and let $x \in M$. $\mathscr{V}$ is attainable from x if and only if $L(\mathscr{V})_x = T_xM$. Furthermore, the interior of $\mathcal{O}^+(x, \mathscr{V})$ is dense in $\mathcal{O}^+(x, \mathscr{V})$.*

The final assertion of the theorem is important as it tells us that the character of the set $\mathcal{O}^+(x, \mathscr{V})$ is not too nasty. For example, it rules out situations like that represented by Figure 5 where there are thin subsets branching off a nice open set.

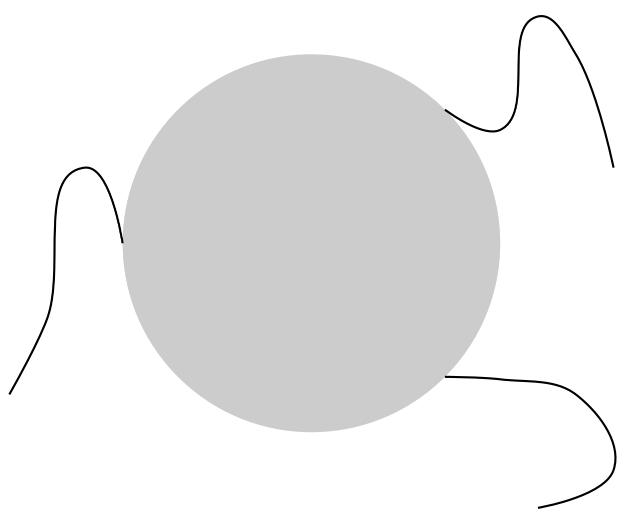

Figure 5. This cannot be the picture for $\mathcal{O}^+(x, \mathscr{V})$

The characterisation of strong attainability requires some non-obvious manipulations with Lie algebras. First let us make a general context for this by recalling some Lie group facts that are at least true for finite-dimensional Lie groups (of which $\mathrm{Diff}(\mathscr{V})$ is most certainly not an example). Thus we let G be a Lie group with H a normal subgroup. We let gH be the coset through $g \in G$, and we denote by G/H the set of cosets. Normality of H is readily seen to imply that the operation $(g_1H)(g_2H) = (g_1g_2)H$ makes G/H into a group, and if H is closed, it is a Lie group. This establishes H as the kernel of the group homomorphism $\pi\colon G \to G/H$. Thus the kernel of the induced Lie algebra homomorphism $T_e\pi\colon T_eG \to T_{eH}(G/H)$ is an ideal.[7] The ***derived algebra*** of a Lie algebra $\mathfrak{g}$ is the Lie subalgebra $\mathfrak{g}'$ of $\mathfrak{g}$ generated by $[u, v]$ for $u, v \in \mathfrak{g}$. Thus $\mathfrak{g}'$ is the subspace generated by elements of $\mathfrak{g}$ of the form

$$[\xi_{i_1}, \xi_{i_2}],\ [\xi_{i_1}, [\xi_{i_2}, \xi_{i_3}]], \dots \qquad \xi_{i_k} \in \mathfrak{g}. \tag{2.2}$$

With this as backdrop, we may expect that the vector fields that generate $\mathrm{Diff}_T(\mathscr{V})$, in the same way that $\mathscr{L}(\mathscr{V})$ generates $\mathrm{Diff}(\mathscr{V})$, should form an ideal. Sussmann and Jurdjevic [1972] argue that this ideal is defined as follows. We let $\mathscr{V}_0$ be the vector fields of the form

$$\sum_{j=1}^{k} \lambda_j X_j \qquad X_1, \dots, X_k \in \mathscr{V},\ \sum_{j=1}^{k} \lambda_j = 0,$$

and let $\mathscr{L}'(\mathscr{V})$ be the derived algebra of $\mathscr{L}(\mathscr{V})$. We then define $\mathscr{I}(\mathscr{V})$ to be generated by vector fields of the form

$$X + Y, \qquad X \in \mathscr{V}_0,\ Y \in \mathscr{L}'(\mathscr{V}).$$

As usual, we define

$$I(\mathscr{V})_x = \{X(x) \mid\ X \in \mathscr{I}(\mathscr{V})\}$$

so that $I(\mathscr{V})$ is a distribution on M, again in an appropriately general sense of the word distribution. One may verify that $\mathscr{I}(\mathscr{V})$ is an involutive family, meaning that $[X, Y] \in \mathscr{I}(\mathscr{V})$ if $X, Y \in \mathscr{I}(\mathscr{V})$. A theorem of Nagano [1966] ensures that this means that $\mathscr{I}(\mathscr{V})$ possesses a maximal integral manifold through any point x, and that the tangent space of this integral manifold at any point y is $I(\mathscr{V})_x$. Nagano's theorem is a generalisation of the classical Frobenius theorem.[8] The picture one should have in mind is that $\mathscr{I}(\mathscr{V})$ is to $\mathcal{O}_T(x, \mathscr{V})$ what $\mathscr{L}(\mathscr{V})$ is to $\mathcal{O}(x, \mathscr{V})$. Indeed, one has the following theorem.

[7] Recall that an ***ideal*** in a Lie algebra $(L, [\cdot, \cdot])$ is a subspace U for which $[u, v] \in U$ for every $u \in U$ and $v \in V$. Often ones writes $[U, V] \subset U$ to characterise an ideal. One readily shows that the kernel of a Lie algebra homomorphism is an ideal, and conversely that every ideal arises in this way.

[8] Let us recall this in our language here. Let D be a distribution of constant rank (i.e., $\dim(D_x)$ is independent of x) and let $\Gamma(D)$ be those vector fields taking values in D. D is ***involutive*** if $[X, Y] \in \Gamma(D)$ for each $X, Y \in \Gamma(D)$. D is ***integrable*** if the maximal integral manifold N through x has the property that $D_y = T_yN$ for each $y \in N$.

Frobenius's theorem: D as above is integrable if and only if it is involutive.

The generalisation provided by Nagano [1966] is essentially that this holds even when $\dim(D_x)$ depends on x. This is a significant generalisation.

2.5 Theorem: *If $\mathscr{V}$ is a family of complete analytic vector fields on M and $x \in M$, then the following statements are true for each $T \in \mathbb{R}$:*

(i) $\mathcal{O}_T(x, \mathscr{V})$ *is an analytic immersed submanifold;*

(ii) for each $y \in \mathcal{O}_T(x, \mathscr{V})$, $T_y(\mathcal{O}_T(x, \mathscr{V})) = I(\mathscr{V})_y$*;*

(iii) M *is the disjoint union of all orbits of* $\mathscr{V}$*.*

Then one has the by now obviously true—at least if there is any order in the world—theorem concerning strong attainability, analogous to Theorem 2.4.

2.6 Theorem: *Let $\mathscr{V}$ be an analytic family of vector fields on M, and let $x \in M$. $\mathscr{V}$ is strongly attainable from x if and only if $I(\mathscr{V})_x = T_x M$. Furthermore, the interior of $\mathcal{O}_T^+(x, \mathscr{V})$ is dense in $\mathcal{O}_T^+(x, \mathscr{V})$.*

2.2. From attainability to accessibility. Most of the hard work for accessibility is contained in the attainability results from the preceding section. However, what we can do is explicitly provide the connection, and in so doing, arrive at fairly easily computable conditions for accessibility and strong accessibility.

First let us deal with accessibility. For a control affine system $\Sigma = (M, \mathscr{F}, U)$ we have previously defined the family of vector fields

$$\mathscr{V}_\Sigma = \Big\{ f_0 + \sum_{a=1}^{m} u^a f_a \;\Big|\; u \in U \Big\}.$$

To this family of vector fields, all of the machinery of Section 2.1 can be applied. However, we wish to see exactly how $\mathscr{V}_\Sigma$ is related to $\mathscr{F}$. In particular, we wish to explore the relationship between $\mathscr{L}(\mathscr{F})$ and $\mathscr{L}(\mathscr{V}_\Sigma)$. To do so, we introduce some simple ideas. We let V be a $\mathbb{R}$-vector space. A subset $A \subset V$ is ***convex*** if $v_1, v_2 \in A$ imply that

$$\{(1-t)v_1 + t v_2 \mid t \in [0,1]\} \subset A.$$

Thus a set is convex when the line connecting any two points in the set lies within the set. If $A \subset V$ is a general subset, a ***convex combination*** of vectors $v_1, \dots, v_k \in A$ is a linear combination of the form

$$\lambda_1 v_1 + \cdots + \lambda_k v_k, \quad \lambda_1, \dots, \lambda_k \geq 0,\ \sum_{\alpha=1}^{l} \lambda_\alpha = 1,\ k \in \mathbb{N}.$$

A set may be verified as being convex if and only if it contains all convex combinations of its points. The ***convex hull*** of a general subset A, denoted $\operatorname{conv}(A)$, is the smallest convex set containing A. One may show that $\operatorname{conv}(A)$ consists of the union of all convex combinations of elements of A. Still in a $\mathbb{R}$-vector space V, an ***affine subspace*** of V is a subset of the form

$$\{v + u \mid u \in U\}$$

for some subspace U. Thus an affine subspace is a "shifted subspace." Given a set $A \subset V$, the ***affine hull*** of A, denoted $\operatorname{aff}(A)$, is the smallest affine subspace of V containing A.

Analogous to the convex hull, one may show that the affine hull is the collection of points of the form

$$\sum_{\alpha=1}^{k} \lambda_\alpha v_\alpha, \quad \lambda_1, \dots, \lambda_k \in \mathbb{R}, \ \sum_{\alpha=1}^{k} \lambda_\alpha = 1, \ v_1, \dots, v_k \in A, \ k \in \mathbb{N}.$$

With these notions at hand, we have the following lemma.

2.7 Lemma: *Let $\Sigma = (M, \mathscr{F}, U)$ be a control affine system and suppose that*

(i) $0 \in \operatorname{conv}(U)$ *and*

(ii) $\operatorname{aff}(U) = \mathbb{R}^m$.

Then $\operatorname{span}_{\mathbb{R}}(\mathscr{F}) = \operatorname{span}_{\mathbb{R}}(\mathscr{V}_\Sigma)$ *which implies that* $\mathscr{L}(\mathscr{F}) = \mathscr{L}(\mathscr{V}_\Sigma)$.

Proof: By definition of $\operatorname{span}_{\mathbb{R}}(\mathscr{V}_\Sigma)$ the inclusion $\operatorname{span}_{\mathbb{R}}(\mathscr{V}_\Sigma) \subset \operatorname{span}_{\mathbb{R}}(\mathscr{F})$ holds. By (i) there exists $\lambda_1, \dots, \lambda_k \in \geq 0$ and $u_1, \dots, u_k \in U$ so that

$$\sum_{j=1}^{k} \lambda_j = 1, \quad 0 = \sum_{j=1}^{k} \lambda_j u_j.$$

Therefore

$$\sum_{j=1}^{k} \lambda_j \Big(f_0 + \sum_{a=1}^{m} u_j^a f_a \Big) = f_0 + \sum_{j=1}^{k} \sum_{a=1}^{m} \lambda_j u_j^a f_a = f_0.$$

Thus $f_0 \in \operatorname{span}_{\mathbb{R}}(\mathscr{V}_\Sigma)$.

Similarly, by (ii), for each $a \in \{1, \dots, m\}$ there exists $\lambda_1, \dots, \lambda_k \in \mathbb{R}$ and $u_1, \dots, u_k \in U$ so that

$$\sum_{j=1}^{k} \lambda_j = 1, \quad \boldsymbol{e}_a = \sum_{j=1}^{k} \lambda_j u_j,$$

where $\boldsymbol{e}_a$, $a \in \{1, \dots, m\}$, is the ath standard basis vector for $\mathbb{R}^m$. Therefore

$$\sum_{j=1}^{k} \lambda_j \Big(f_0 + \sum_{a=1}^{m} u_j^a f_a \Big) = f_0 + \sum_{j=1}^{k} \sum_{b=1}^{m} \lambda_j u_j^b f_b = f_0 + f_a.$$

Thus $f_0 + f_a \in \operatorname{span}_{\mathbb{R}}(\mathscr{V}_\Sigma)$, showing that $f_a \in \operatorname{span}_{\mathbb{R}}(\mathscr{V}_\Sigma)$, $a \in \{1, \dots, m\}$. Thus we have shown that the inclusion $\operatorname{span}_{\mathbb{R}}(\mathscr{F}) \subset \operatorname{span}_{\mathbb{R}}(\mathscr{V}_\Sigma)$ also holds. ■

With this as motivation, let us call a subset $U \subset \mathbb{R}^m$ ***almost proper*** if it has properties (i) and (ii) of the lemma. If $0 \in \operatorname{int}(\operatorname{conv}(U))$ then U is ***proper***.

Now we may state the following result, derived from Theorems 2.1 and 2.4, characterising accessibility.

2.8 Theorem: *Let $\Sigma = (M, \mathscr{F}, U)$ be an analytic control affine system with U almost proper. Then Σ is accessible from x if and only if $L(\mathscr{F})_x = T_x M$.*

Proof: Let us denote by $\mathcal{O}(x, \mathscr{F})$ the $\mathscr{F}$-orbit through x. Note that the vector fields $f_0, f_1, \dots, f_m$ are tangent to $\mathcal{O}(x, \mathscr{F})$. Since a controlled trajectory (c, u) has the property that c is an absolutely continuous curve for which

$$c'(t) \in \operatorname{span}_{\mathbb{R}}(f_0(c(t)), f_1(c(t)), \dots, f_m(c(t))) \quad \text{a.e.},$$

it follows that if $c(0) = x$ then $c(t) \in \mathcal{O}(x, \mathscr{F})$. Thus $\mathcal{R}_\Sigma(x) \subset \mathcal{O}(x, \mathscr{F})$. In particular, if Σ is locally accessible we must have $T_x(\mathcal{O}(x, \mathscr{F})) = T_x M$. From Theorem 2.1 this implies that $L(\mathscr{F})_x = T_x M$.

Now suppose that $L(\mathscr{F})_x = T_x M$. By Lemma 2.7 this implies that $L(\mathscr{V}_\Sigma)_x = T_x M$. Now, since piecewise constant controls $u\colon [0, T] \to U$ are measurable, it follows that $\mathcal{O}^+(x, \mathscr{V}_\Sigma) \subset \mathcal{R}_\Sigma(x)$. From Theorem 2.4 this means that $\operatorname{int}(\mathcal{R}_\Sigma(x)) \neq \emptyset$. ■

Note that for C^∞ systems, the condition that $L(\mathscr{F})_x = T_x M$ is only sufficient for accessibility. There are C^∞ systems which are accessible, but which violate this condition. They are crazy, however, as is always the case for things that are C^∞ but not analytic.

For strong accessibility, we need to construct the analogue of $\mathscr{L}(\mathscr{F})$. More precisely, we need to find that object which relates to $\mathscr{I}(\mathscr{V}_\Sigma)$ in the same way that $\mathscr{L}(\mathscr{F})$ relates to $\mathscr{L}(\mathscr{V}_\Sigma)$. The following result begins this characterisation.

2.9 Lemma: *Let $\Sigma = (M, \mathscr{F} = \{f_0, f_1, \dots, f_m\}, U)$ be a control affine system. Let $\mathscr{L}_0(\mathscr{F})$ be the smallest subalgebra of $\Gamma(TM)$ containing $\{f_1, \dots, f_m\}$ and which is invariant under f_0, i.e., $[f_0, X] \in \mathscr{L}_0(\mathscr{F})$ for each $X \in \mathscr{L}_0(\mathscr{F})$. The following statements hold:*

(i) $\mathscr{L}_0(\mathscr{F})$ is generated as a $\mathbb{R}$-vector space by vector fields of the form

$$[f_{a_1}, [f_{a_2}, \cdots, [f_{a_{k-1}}, f_a]]], \quad a_1, \dots, a_{k-1} \in \{0, 1, \dots, m\}, \; a \in \{1, \dots, m\}; \qquad (2.3)$$

(ii) if U is almost proper then $\mathscr{L}_0(\mathscr{F}) = \mathscr{I}(\mathscr{V}_\Sigma)$.

Proof: (i) Clearly $f_1, \dots, f_m \in \mathscr{L}_0(\mathscr{F})$. Also, since $\mathscr{L}_0(\mathscr{F})$ is involutive and invariant under f_0, the vector fields $[f_{a_1}, f_a]$, $a_1 \in \{0, 1, \dots, m\}$, $a \in \{1, \dots, m\}$, are in $\mathscr{L}_0(\mathscr{F})$. Continuing in this way one readily sees that each of the vector fields of the form (2.3) is in $\mathscr{L}_0(\mathscr{F})$. Thus the vector fields (2.3) must be contained in any set of generators for $\mathscr{L}_0(\mathscr{F})$. The lemma follows since by definition $\mathscr{L}_0(\mathscr{F})$ is the smallest subalgebra containing these generators.

(ii) If $u \in U$ let us write

$$f_u = f_0 + \sum_{a=1}^{m} u^a f_a \in \Gamma^\omega(TM).$$

Since U is almost proper we have $\operatorname{span}_{\mathbb{R}}(\mathscr{F}) = \operatorname{span}_{\mathbb{R}}(\mathscr{V}_\Sigma)$, meaning that the derived algebras of $\mathscr{L}(\mathscr{F})$ and $\mathscr{L}(\mathscr{V}_\Sigma)$ agree: $\mathscr{L}'(\mathscr{F}) = \mathscr{L}'(\mathscr{V}_\Sigma)$. Since $\mathscr{L}'(\mathscr{F})$ is the subalgebra generated by the vector fields

$$[f_{a_1}, f_{a_2}], \; [f_{a_1}, [f_{a_2}, f_{a_3}]], \dots \qquad f_{a_k} \in \mathscr{F},$$

the lemma will be proved if we can show that $\operatorname{span}_{\mathbb{R}}(\mathscr{V}_{\Sigma,0}) = \operatorname{span}_{\mathbb{R}}(f_1, \dots, f_m)$. A typical element of $\mathscr{V}_{\Sigma,0}$ looks like

$$\sum_{j=1}^{k} \lambda_j \Big(f_0 + \sum_{a=1}^{m} u_j^a f_a \Big) = \sum_{j=1}^{k} \lambda_j \sum_{a=1}^{m} u_j^a f_a, \qquad \sum_{j=1}^{k} \lambda_j = 0.$$

Thus we obviously have $\operatorname{span}_{\mathbb{R}}(\mathscr{V}_{\Sigma,0}) \subset \operatorname{span}_{\mathbb{R}}(f_1, \dots, f_m)$. Since U is almost proper we have $\operatorname{aff}(U) = \mathbb{R}^m$. Thus the subspace part of $\operatorname{aff}(U)$ is also $\mathbb{R}^m$. This means that for any $a \in \{1, \dots, m\}$ there exists $\lambda_1, \dots, \lambda_k \in \mathbb{R}$ and $u_1, \dots, u_k \in U$ so that

$$\sum_{j=1}^{k} \lambda_j = 1, \quad \sum_{j=1}^{k} \lambda_j u_j = \boldsymbol{e}_a.$$

Therefore

$$\sum_{j=1}^{k} \sum_{b=1}^{m} \lambda_j u_j^b f_b = f_a,$$

showing that $\operatorname{span}_{\mathbb{R}}(f_1, \dots, f_m) \subset \operatorname{span}_{\mathbb{R}}(\mathscr{V}_{\Sigma,0})$, and so proving the lemma. ■

As usual, we denote

$$L_0(\mathscr{F})_x = \{X(x) \mid X \in \mathscr{L}_0(\mathscr{F})\}.$$

With this characterisation of $\mathscr{I}(\mathscr{V}_\Sigma)$, one may now prove the following result from Theorem 2.6 and Nagano's theorem concerning involutive families of vector fields.

2.10 Theorem: *Let $\Sigma = (M, \mathscr{F}, U)$ be an analytic control affine system with U almost proper. Then Σ is strongly accessible from x if and only if $L_0(\mathscr{F})_x = T_x M$.*

Proof: Let us construct a control-affine $\tilde{\Sigma} = (\tilde{M}, \tilde{\mathscr{F}} = \{\tilde{f}_0, \tilde{f}_1, \dots, \tilde{f}_m\}, U)$ with $\tilde{M} = M \times \mathbb{R}$, $\tilde{f}_0(x,t) = f_0(x) + \frac{\partial}{\partial t}$, $\tilde{f}_a(x,t) = f_a(x)$, $a \in \{1, \dots, m\}$. We make the following easily verified assertions about $\tilde{\Sigma}$:

1. if Σ is strongly accessible from x then $\tilde{\Sigma}$ is accessible from $(x, 0)$;
2. $L(\tilde{\mathscr{F}})_x = L_0(\mathscr{F})_x + \operatorname{span}_{\mathbb{R}}(f_0(x) + \frac{\partial}{\partial t})$.

Now suppose that Σ is strongly accessible from x by 1. Then $\tilde{\Sigma}$ is accessible from $(x, 0)$ so $L(\mathscr{F})_x = T_{(x,0)}\tilde{M}$ by Theorem 2.8. Therefore, by 2, $L_0(\mathscr{F})_x = T_x M$ since $\operatorname{span}_{\mathbb{R}}(f_0(x) + \frac{\partial}{\partial t})$ is complementary to $T_x M$.

Now suppose that $L_0(\mathscr{F})_x = T_x M$. By Lemma 2.9 this means that $I(\mathscr{V}_\Sigma)_x = T_x M$. Since piecewise constant controls are measurable, by Theorem 2.6 it follows that $\mathcal{O}_T^+(x_0, \mathscr{V}_\Sigma) \subset \mathcal{R}_\Sigma(x, T)$. Also from Theorem 2.6, we therefore conclude that Σ is strongly accessible. ■

Note that we have shown in this section that there are computable (at least in terms of differentiations and linear algebra) necessary and sufficient conditions for accessibility and strong accessibility, at least for analytic systems with a sufficiently nice control set. As we shall see, things are not in such good shape for controllability.

3. Discussions surrounding controllability

In this section we survey the grim landscape of controllability results for nonlinear systems. As we shall see, the extent of our knowledge, while having some substance, is embarrassingly incomplete. Certainly this is not due to a lack of effort, as many people, some of them smart, have worked on the problem of local controllability. A very incomplete list of papers on the subject is the following: [Aeyels 1984, Agrachev 1999, Bacciotti and Stefani 1983, Basto-Gonçalves 1985, Basto-Gonçalves 1998, Bianchini and Stefani 1984, Bianchini and Stefani 1986, Bianchini and Stefani 1993, Boltyanskiĭ 1981, Haynes and Hermes 1970, Hermes 1976a, Hermes 1976b, Hermes 1977, Hermes 1982, Hermes and Kawski 1987, Kawski 1987, Kawski 1990, Kawski 1991, Knobloch 1977, Knobloch and Wagner 1984, Petrov 1977, Stefani 1985, Stefani 1986, Sussmann 1978, Sussmann 1983a, Sussmann 1983b, Sussmann 1987, Vârsan 1974, Warga 1985].

For simplicity, let us assume in this section that "controllability" means "small-time local controllability."

3.1. Controllability and feedback-invariance. If one is to "solve" the problem of nonlinear controllability, one might start by defining the terms by which you will negotiate with the problem. This is what we do here. It is convenient to be able to regard control affine systems as a "category." This approach is taken by Elkin [1998] for the purposes of classifying control affine systems by "equivalence." For us, it will simply serve as a useful way of talking about feedback transformations. A ***category***, roughly, is a collection of ***objects*** and a collection of maps between these objects, called ***morphisms***, which preserve the structure of the objects. For example, one has the category of $\mathbb{R}$-vector spaces whose objects are (of course) $\mathbb{R}$-vector spaces, and whose morphisms are $\mathbb{R}$-linear maps. We denote by CAS the category whose objects are analytic control affine systems $\Sigma = (M, \mathscr{F}, U)$. For simplicity in the defining of morphisms, let us assume that $U = \mathbb{R}^m$. More general control sets are allowable, but the definitions need to be additionally complicated, obscuring their essential geometry. Suppose that we have two objects $\Sigma = (M, \mathscr{F} = \{f_0, f_1, \dots, f_m\}, \mathbb{R}^m)$ and $\tilde{\Sigma} = (\tilde{M}, \tilde{\mathscr{F}} = \{\tilde{f}_0, \tilde{f}_1, \dots, \tilde{f}_{\tilde{m}}\}, \mathbb{R}^{\tilde{m}})$. We let $\mathrm{L}(\mathbb{R}^m; \mathbb{R}^{\tilde{m}})$ denote the set of linear maps from $\mathbb{R}^m$ to $\mathbb{R}^{\tilde{m}}$. A **CAS *morphism*** sending Σ to $\tilde{\Sigma}$ is a triple $(\psi, \lambda_0, \Lambda)$ with the following properties:

1. $\psi\colon M \to N$ is an analytic mapping;
2. $\lambda_0\colon M \to \mathbb{R}^{\tilde{m}}$ and $\Lambda\colon M \to \mathrm{L}(\mathbb{R}^m; \mathbb{R}^{\tilde{m}})$ are analytic mappings satisfying
   (a) $T_x\psi(f_a(x)) = \Lambda^\alpha_a(x)\tilde{f}_\alpha(\psi(x))$, $a = 1, \dots, m$ and
   (b) $T_x\psi(f_0(x)) = \tilde{f}_0(\psi(x)) + \lambda^\alpha_0(x)\tilde{f}_\alpha(\psi(x))$.

An essential feature of this class of morphisms is then given by the following straightforward result of Elkin [1998]. If c is a curve on M and $\psi\colon M \to \tilde{M}$ is a map, the curve $\psi \circ c$ on $\tilde{M}$ is written as c_ψ.

3.1 Proposition: *If $(\psi, \lambda_0, \Lambda)$ is a morphism in CAS which maps $\Sigma = (M, \mathscr{F}, \mathbb{R}^m)$ to $\tilde{\Sigma} = (\tilde{M}, \tilde{\mathscr{F}}, \mathbb{R}^{\tilde{m}})$ and if (c, u) is a controlled trajectory for Σ, then $(c_\psi, \tilde{u})$ is a controlled trajectory for $\tilde{\Sigma}$ where $\tilde{u}(t) = \lambda_0(c(t)) + \Lambda(c(t))u(t)$.*

Conversely, suppose that $\psi\colon M \to \tilde{M}$ *is a smooth mapping which has the property that for every controlled trajectory* (c, u) *of* Σ *there exists an admissible input* $\tilde{u}$ *so that* $(c_\psi, \tilde{u})$ *is a controlled trajectory for* $\tilde{\Sigma}$*. Then there exists smooth mappings* $\lambda_0\colon M \to \mathbb{R}^{\tilde{m}}$ *and* $\Lambda\colon M \to \mathrm{L}(\mathbb{R}^m; \mathbb{R}^{\tilde{m}})$ *so that* $(\psi, \lambda_0, \Lambda)$ *is a* CAS *morphism sending* Σ *to* $\tilde{\Sigma}$*.*

The punchline is that a morphism $(\psi, \lambda_0, \Lambda)$ sends the control affine system $(M, \mathscr{F}, \mathbb{R}^m)$ to the control affine system $\tilde{\Sigma} = (\tilde{M}, \tilde{\mathscr{F}}, \mathbb{R}^{\tilde{m}})$ where

$$T_x\psi(f_0(x)) = \tilde{f}_0(\psi(x)) + \sum_{\alpha=1}^{\tilde{m}} \lambda_0^\alpha(x)\tilde{f}_\alpha(\psi(x))$$

and

$$T_x\psi(f_a(x)) = \sum_{\alpha=1}^{\tilde{m}} \big(\Lambda_a^\alpha(x) - \lambda_0^\alpha(x)\big)\tilde{f}_\alpha(\psi(x)).$$

Alternatively, one can think of CAS morphisms as transformations of the state and control of the form

$$(x, u) \mapsto (\psi(x), \lambda_0(x) + \Lambda(x)u).$$

Interesting cases include the following.

1. ψ is a diffeomorphism, $\lambda_0 = 0$, and $\Lambda(x) = \mathrm{id}_{\mathbb{R}^m}$ for each $x \in M$. This amounts to a change of coordinates.
2. $M = \tilde{M}$, $\psi = \mathrm{id}_M$, $\lambda_0 = 0$, and $\Lambda\colon M \to \mathrm{L}(\mathbb{R}^m; \mathbb{R}^m)$ takes values in $GL(m; \mathbb{R})$. This amounts to a change of basis for the control vector fields.
3. $M = \tilde{M}$, $\psi = \mathrm{id}_M$, and $\Lambda(x) = \mathrm{id}_{\mathbb{R}^m}$. This amounts redefining the inputs to eliminate some terms in the drift vector field that are displeasing. For example, when

$$\mathrm{span}_{\mathbb{R}}(f_1(x), \dots, f_m(x)) = T_xM$$

for each $x \in M$, then a morphism of this type will reduce the system to one that is driftless.

Note that the first type of morphism is a pure coordinate change, with the controls left alone, whereas the second two are purely algebraic operations on the controls. Elkin [1998] explores when a given morphism may be regarded as a composition of two morphisms, each being of one of the previous two types.

A special kind of morphism is an ***isomorphism***. This establishes an equivalence between two objects in the category. A CAS morphism $(\psi, \lambda_0, \Lambda)$ is an isomorphism when ψ is a diffeomorphism. This then establishes an exact correspondence between the controlled trajectories of Σ and those of $\tilde{\Sigma}$. It is clear then that if there is a CAS isomorphism between two control affine systems, they will have the same controllability properties. That is to say, controllability is a "feedback-invariant" property. It seems reasonable then to seek *conditions* for controllability that are also feedback-invariant.

3.2. Setting up a framework to solve the problem. The previous section suggests that we seek feedback-invariant controllability conditions. What form should such conditions take? This is addressed in the introduction of the early paper on controllability by Sussmann [1978]. His approach is to say that one should attack controllability inductively on the order of the derivatives involved. This is just like one might do in seeking conditions for whether a given point is a minimum for a $\mathbb{R}$-valued function. In this case

0. there are no zeroth-order conditions (one cannot tell whether a function is at a minimum merely by knowing its value),
1. the first-order necessary condition is that the derivative vanish, and there are no first-order sufficient conditions,
2. the second-order sufficient condition is that the Hessian be positive-definite, and the second-order necessary condition is that the Hessian be positive-semidefinite,
3. I am guessing that there are higher-order conditions known, but I do not know them offhand...

Sussmann proposes doing the same thing for local controllability. The idea is that for each $k \geq 0$ the set of control affine systems (actually Sussmann worked in a slightly different context, but never mind) can be broken into three classes: (1) those that can be shown to be controllable by using derivatives of vector fields in $\mathscr{F}$ up to order k, (2) those that can be shown to be uncontrollable by using derivatives of vector fields in $\mathscr{F}$ up to order k, and (3) those whose controllability cannot be decided by using derivatives of vector fields in $\mathscr{F}$ up to order k.

This is an interesting idea, but it leaves open what it means to be able to "decide by using derivatives of vector fields in $\mathscr{F}$ up to order k." Let us set this matter aside for a moment, (falsely) supposing that we have a way of providing a means to make these decisions. What one then wants to do is come up with two computable conditions, one being a sufficient condition, the other being a necessary condition. These conditions need to be sharp, by which one means that one should be able to prove that if a system satisfies neither the sufficient nor the necessary condition, then it is not possible to ascertain the controllability of the system using k derivatives. At some orders, the necessary condition will be vacuous. That is to say, it is possible for certain k's that it is not possible to say that a system is not controllable using derivatives up to order k, except to use the already existing lower-order necessary conditions. In such cases, we shall say that there are no ***kth-order obstructions to controllability***. Again, this is vague, and part of a solution to the controllability problem will be an understanding of how to clarify these ideas.

As we shall see in Section 3.4, Sussmann [1978] deals with this in the case when $k = 0$, and the $k = 1$ case is also resolved. However, for higher-order conditions, it is not clear how to proceed. We suggest an approach for second-order conditions in Section 3.5.

3.3. Technology for providing controllability conditions. Let us now turn to the matter of how to proceed to *get* controllability conditions. In Section 2 we saw that the Lie bracket played a crucial rôle in the theory of accessibility. The same is true for controllability, although it not so obvious why this should be the case. Some ray of hope comes from another theorem from the paper of Nagano [1966]. Suppose that we have families of vector fields, $\mathscr{X}$ on M and $\mathscr{Y}$ on N. Also suppose that for $x \in M$, $M = \mathcal{O}(x, \mathscr{X})$ and that for $y \in N$, $N = \mathcal{O}(y, \mathscr{Y})$. We also assume that there is an isomorphism $L\colon T_xM \to T_yN$ and a bijection $\kappa\colon \mathscr{X} \to \mathscr{Y}$ (in particular, $\dim(M) = \dim(N)$ and $\mathscr{X}$ and $\mathscr{Y}$ have the same cardinality), and that this isomorphism has the property that for any $X_1, \dots, X_k \in \mathscr{X}$ we have

$$L([X_1, [X_2, \cdots, [X_{k-1}, X_k]]](x)) = [\kappa(X_1), [\kappa(X_2), \cdots, [\kappa(X_{k-1}), \kappa(X_k)]]](y).$$

This means that the bracket relations at x for $\mathscr{X}$ are the same as those at y for $\mathscr{Y}$. Nagano shows that, if the families $\mathscr{X}$ and $\mathscr{Y}$ are analytic (of course), then there is a diffeomorphism ψ from a neighbourhood $\mathcal{M}$ of x to a neighbourhood $\mathcal{N}$ of y which sends $X \in \mathscr{X}|\mathcal{M}$ to $\kappa(X) \in \mathscr{Y}|\mathcal{N}$. In particular, the trajectories defined by the two families of vector fields are identical up to the diffeomorphism ψ, at least close to the points x and y. The idea of this is then that one can exactly characterise the properties of a family of vector fields near x by looking only at the Lie brackets of these vector fields *evaluated at* x. Motivated by this, Sussmann [1983a] sets about providing a systematic structure for analysing controllability using series involving Lie brackets. This culminates in the quite general controllability results of [Sussmann 1987], which were further generalised by Bianchini and Stefani [1993].

In [Bianchini and Stefani 1993], the useful idea of a control variation is introduced. We let $\Sigma = (M, \mathscr{F}, U)$ be a control affine system with U proper and let $x \in M$ be an equilibrium point for f_0. Roughly speaking, a ***control variation*** at $x \in M$ is a vector $v \in T_xM$ for which there exists a one-parameter family u_ϵ of controls with $u_0 = 0$ and so that the controlled trajectory (c_ϵ, u_ϵ) with $c_\epsilon(0) = x$ has the property that

$$c_\epsilon(\tau(u_\epsilon)) = x + \epsilon v + \text{h.o.t.s.}$$

Control variations with "nice" properties are handed the monicker "regular." Essentially, the control leading to a regular variation should be embedded in a family of such controls. The ***variational cone*** is the smallest cone containing all regular control variations. Bianchini and Stefani show that if the variational cone is T_xM, then Σ is STLC at x. (The state of the necessity of this condition is unknown to the author.) They then illustrate how certain regular variations can be constructed using Lie brackets at x. In this way, they obtain results more general than those of Sussmann [1987].

3.4. Some known conditions. Let us turn now to a review of all that is known about the program outlined in Section 3.2. As mentioned above, only the cases $k \in \{0, 1\}$ have been exhaustively treated. Sadly, as we shall see, these cases are actually quite simple. The zeroth-order case is intuitively clear, and the first-order case is essentially uninteresting as there are no obstructions to controllability at first-order.

The zeroth-order case

Let us consider the zeroth-order case as treated by Sussmann [1978]. For a control affine system $\Sigma = (M, \mathscr{F}, U)$ and for $x \in M$ let us denote

$$\mathscr{V}_\Sigma(x)(x) = \left\{ f_0(x) + \sum_{a=1}^m u^a f_a(x) \;\middle|\; u \in U \right\} \subset T_xM.$$

Let us say that a control affine system Σ is $\boldsymbol{STLC_0}$ at x if for every control affine system $\tilde{\Sigma} = (M, \tilde{\mathscr{F}}, \tilde{U})$ for which $\tilde{\mathscr{F}}_{\tilde{U}}(x) = \mathscr{V}_\Sigma(x)(x)$, $\tilde{\Sigma}$ is controllable if and only if Σ is controllable. Sussmann [1978] then (essentially, as his setup is slightly different) proves the following result.

3.2 Theorem: *Let $\Sigma = (M, \mathscr{F}, U)$ be a control affine system and let $x \in M$. The following statements hold.*

(i) Σ is STLC from x if $0 \in \operatorname{int}(\operatorname{conv}(\mathscr{V}_\Sigma(x)(x)))$.

(ii) Σ *is not STLC from* x *if* $0 \notin \operatorname{conv}(\mathscr{V}_\Sigma(x)(x))$*.*

Furthermore, Σ *is STLC*$_0$ *if and only if it satisfies either (i) or (ii).*

The idea of the theorem is clear. Let us make some remarks indicating the central ideas.

3.3 Remarks: 1. For the sufficient condition (i), the system is fully actuated and the control set has the property that it is possible to overcome the drift dynamics at x via controls. Therefore, one may make a feedback transformation which turns the system into essentially a fully actuated driftless system. Such systems are trivially controllable. This is intuitive, of course, but it can be fairly easily turned into a complete argument.

2. For the necessary condition (ii), the idea is that if $0 \notin \operatorname{conv}(\mathscr{V}_\Sigma(x)(x))$ then at x the drift vector field dynamics will dominate the controls, and so for small times you will essentially move in the direction specified by the drift.

3. The above two remarks assume that the control set U is proper. If U is not proper and if the necessary condition is not met, then the system can be uncontrollable in ways other than that suggested in 2.

4. Note that if $f_0(x) = 0$ (i.e., we are at an equilibrium point for f_0) and if the control set is proper, then the necessary condition is always satisfied. This means that for such systems, one cannot say that the system is uncontrollable using zeroth-order information. •

The first-order case

The first-order characterisation we give is due to Bianchini and Stefani [1984]. In order to move from zeroth-order to first-order conditions, one should naturally assume that the zeroth-order necessary condition is satisfied. As we saw in our Remark 3.3–4, this necessary condition is always satisfied if $f_0(x) = 0$ and if U is proper. In fact, in this case we have

$$0 \in \operatorname{int}_{\operatorname{aff}(\mathscr{V}_\Sigma(x)(x))}(\operatorname{conv}(\mathscr{V}_\Sigma(x)(x))), \tag{3.1}$$

where int_A means the interior relative to the induced topology on the subset A. Let us make the assumption, as Bianchini and Stefani do, that (3.1) holds. This assumption, note, is generally stronger than the necessary condition (ii) of Theorem 3.2 since the latter only asks that $0 \in \operatorname{conv}(\mathscr{V}_\Sigma(x)(x))$. Nevertheless, we make this stronger assumption, purely for convenience, but with the justification that it holds in the "standard" case. With this assumption, we may as well also assume that $f_0(x) = 0$ since $f_0(x)$ will be a convex combination of the controlled tangent vectors at x.

Moving on... Let us partition $\mathscr{F}$ into two sets $\mathscr{F}^0(x)$ and $\mathscr{F}^1(x)$ defined by

$$\mathscr{F}^0(x) = \{f_a \in \mathscr{F} \mid\ f_a(x) = 0\}, \quad \mathscr{F}^1(x) = \mathscr{F} \setminus \mathscr{F}^0(x).$$

Our assumptions above ensure that $f_0 \in \mathscr{F}^0(x)$. We inductively define subsets of T_xM by

$$\begin{aligned} L(\mathscr{F})_x^{(0)} &= \{f_a(x) \mid\ f_a \in \mathscr{F}\} \\ L(\mathscr{F})_x^{(1)} &= \{[f_a, f_b](x) \mid\ f_a, f_b \in \mathscr{F}\} \\ &\ \vdots \\ L(\mathscr{F})_x^{(k)} &= \big\{[X, Y](x) \bigm|\ X \in L(\mathscr{F})_x^{(k_1)},\ Y \in L(\mathscr{F})_x^{(k_2)},\ k_1 + k_2 = k + 1\big\} \\ &\ \vdots \end{aligned}$$

Next, for $f_a \in \mathscr{F}^0(x)$, define a linear map $L_{f_a} \colon T_xM \to T_xM$ by

$$L_{f_a}(v_x) = [f_a, V](x),$$

where V is any vector field having the property that $v_x = V(x)$. A quick peek at the coordinate formula (2.1) for the Lie bracket will convince you that this definition does not depend on the choice of V, and that the indicated map is indeed linear. We now define $C(\mathscr{F})_x^{(1)}$ to be the subspace of T_xM generated by the vectors

$$L_{f_{a_1}} \circ \cdots \circ L_{f_{a_k}}(X), \quad f_{a_1}, \ldots, f_{a_k} \in \mathscr{F}^0(x),\ X \in L(\mathscr{F})_x^{(0)} \cup L(\mathscr{F})_x^{(1)},\ k \in \mathbb{N}.$$

As with Theorem 3.2, we need some way of saying that a controllability condition is the best possible. To this end, let us say that two control affine systems $\Sigma = (M, \mathscr{F} = \{f_0, f_1, \ldots, f_m\}, U)$ and $\tilde{\Sigma} = (M, \tilde{\mathscr{F}} = \{\tilde{f}_0, \tilde{f}_1, \ldots, \tilde{f}_m\}, \tilde{U})$ are ***1-equivalent at x*** if $\tilde{U} = U$ and if the 1-jets of $\tilde{f}_a$ and f_a are equal for $a \in \{0, 1, \ldots, m\}$.[9] Let us say that Σ is ***STLC$_1$*** at x if every control affine system that is 1-equivalent to Σ at x is STLC if and only if Σ is STLC. Note that if Σ is STLC but *not* STLC$_1$, then there is a control affine system Σ_{nc}, 1-equivalent to Σ at x, and with the property that Σ_{nc} is not STLC at x. That is to say, the controllability of an STLC but not STLC$_1$ system is not ascertainable at first-order. Also, if Σ is not STLC at x, then any system 1-equivalent to Σ at x is not STLC$_1$.

The following theorem characterises controllability to first-order.

3.4 Theorem: *Let $\Sigma = (M, \mathscr{F}, U)$ be a control affine system with U proper and $f_0(x) = 0$ for $x \in M$. Σ is STLC$_1$ at x if and only if $C(\mathscr{F})_x^{(1)} = T_xM$. In particular,*

(i) if $C(\mathscr{F})_x^{(1)} = T_xM$ then Σ is STLC from x, and

(ii) if Σ is not STLC at x, then there is no system, 1-equivalent to Σ, that is STLC at x.

Let us make a few points about this theorem.

3.5 Remarks: 1. The reader should verify that the above theorem shows that a linear system satisfying the Kalman rank condition is STLC from 0.

2. The idea of the theorem is that the condition that $C(\mathscr{F})_x^{(1)} = T_xM$ is the only first-order condition that can ensure controllability. That is, every other first-order condition must be implied by it. In particular, the first-order result of Sussmann [1978], when put into the control affine context, is implied by Theorem 3.4.

[9] That is to say, the value of f_a and $\tilde{f}_a$ are equal at x, and the value of their first-derivatives are equal at x.

3. There are no 1st-order obstructions to controllability. That is, if a system is uncontrollable, it is not possible to tell this by looking only at the 1-jet of the system (provided, of course, that the zeroth-order necessary condition of Theorem 3.2 is met).
4. The condition that $C(\mathscr{F})_x^{(1)} = T_xM$ is not *obviously* feedback-invariant. However, it is feedback-invariant. What would be interesting would be to provide a characterisation that is more obviously feedback-invariant. •

The second-order case: a sufficient condition and a single-input necessary condition

Now let us give a second-order condition, sort of due to Sussmann [1987], but relying on some generalisations of Bianchini and Stefani [1993]. We do not state the conditions in their full glorious generality, as this generality needs notions of dilations and weights that are painful to resurrect.

We consider a control affine system $\Sigma = (M, \mathscr{F} = \{f_0, f_1, \dots, f_m\}, U)$ with $x \in M$ an equilibrium point for f_0 and with U proper. Let us partition $\mathscr{F}$ as $\mathscr{F}^0(x)$ and $\mathscr{F}^1(x)$ as above. We define $C(\mathscr{F})_x^{(2)}$ as the subspace of T_xM generated by

$$L_{f_{a_1}} \circ \dots \circ L_{f_{a_k}}(X), \quad f_{a_1}, \dots, f_{a_k} \in \mathscr{F}^0(x),\ X \in L(\mathscr{F})_x^{(0)} \cup L(\mathscr{F})_x^{(1)} \cup L(\mathscr{F})_x^{(2)}\ \ k \in \mathbb{N}.$$

A system Σ is ***second-order neutralisable*** at x if

$$[f_1, [f_0, f_1]](x) + \dots + [f_m, [f_0, f_m]](x) \in C(\mathscr{F})_x^{(1)}.$$

The following theorem gives a sufficient condition for a system to be STLC. As stated above, this result is a consequence of more general results given in [Sussmann 1987] and [Bianchini and Stefani 1993]. We also throw in a necessary condition, valid for single-input systems, which first appeared in [Sussmann 1983a].

3.6 Theorem: *Let $\Sigma = (M, \mathscr{F}, U)$ be a control affine system with U proper. Suppose that for $x \in M$, $f_0(x) = 0$ and that*

(i) $C(\mathscr{F})_x^{(2)} = T_xM$ and

(ii) Σ is second-order neutralisable at x.

Then Σ is STLC at x.

If $m = 1$ and Σ is not second-order neutralisable at x then Σ is not STLC at x.

Let us probe this result a little with some examples.

3.7 Examples: 1. First, we remark that the necessary condition of Theorem 3.6 explains why the system of Example 1.2–1 is not STLC. Indeed, for that system one computes

$$[f_1, [f_0, f_1]](0, 0) = \begin{bmatrix} 0 \\ -2 \end{bmatrix}.$$

One also computes $C(\mathscr{F})_{(0,0)}^{(1)} = \operatorname{span}_{\mathbb{R}}((1, 0))$, and so $[f_1, [f_0, f_1]](0, 0) \notin C(\mathscr{F})_{(0,0)}^{(1)}$. Thus Theorem 3.6 tells us that the system is not STLC.

The problem, intuitively, is that the bracket $[f_1, [f_0, f_1]]$ is "quadratic" in the control vector field f_1. Thus, no matter if you go forwards or backwards with the control, the direction of this bracket cannot be changed. Therefore, one can expect that if such a bracket is essential to obtaining an accessible system, it will cause problems with controllability.

Sadly, this intuition does not extend to the multiple-input case.

2. Now let us look at an example where the hypotheses of Theorem 3.6 do *not* hold, but that is controllable. We work with $M = \mathbb{R}^3$, take $m = 2$, and consider the control equations

$$\begin{aligned}\dot{x} &= yz \\ \dot{y} &= -xz + u_1 \\ \dot{z} &= -u_1 + u_2.\end{aligned}$$

We let U be any bounded proper set. We have

$$f_0 = \begin{bmatrix} yz \\ -xz \\ 0 \end{bmatrix}, \quad f_1 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \quad f_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

We then compute some brackets:

$$[f_0, f_1] = \begin{bmatrix} y - z \\ -x \\ 0 \end{bmatrix}, \quad [f_0, f_2] = \begin{bmatrix} -y \\ x \\ 0 \end{bmatrix}, \quad [f_1, f_2] = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

$$[f_1, [f_0, f_1]] = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \quad [f_1, [f_0, f_2]] = [f_2, [f_0, f_1]] = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \quad [f_2, [f_0, f_2]] = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

One can readily check that all of these brackets, along with the input vector fields themselves, span $T_{(0,0,0)}\mathbb{R}^3$, so the system is accessible. Note that all first-order brackets vanish at $(0, 0, 0)$. From this we deduce that $C(\mathscr{F})^{(1)}_{(0,0,0)} = \operatorname{span}_{\mathbb{R}}(f_1(0, 0, 0), f_2(0, 0, 0))$. Next we note that

$$[f_1, [f_0, f_1]](0, 0, 0) + [f_2, [f_0, f_2]](0, 0, 0) \notin \operatorname{span}_{\mathbb{R}}(f_1(0, 0, 0), f_2(0, 0, 0)).$$

Therefore, the hypotheses of Theorem 3.6 are not met. However, the system *is* controllable. . .

3. We consider essentially the same system as in the previous example, except we now make a change of basis for the input vector fields. We now take

$$\tilde{f}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \tilde{f}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

noting that $\tilde{f}_1 = f_1 + f_2$ and $\tilde{f}_2 = f_2$. Thus this is essentially the same system as the previous example, except that we have made a simple feedback transformation. We still determine that all first-order brackets vanish at $(0, 0, 0)$, but now, the second-order brackets are

$$[f_1, [f_0, f_1]] = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad [f_1, [f_0, f_2]] = [f_2, [f_0, f_1]] = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \quad [f_2, [f_0, f_2]] = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Thus the system now satisfies the hypotheses of Theorem 3.6, and so is STLC. Therefore, the sufficient conditions of Theorem 3.6 are not necessary in the multi-input case. •

The "problem" with the notion of second-order neutralisability is that it is not feedback-invariant, as we have seen with the last two examples. Let us now turn to understanding a feedback-invariant representation of the obstruction to controllability offered by systems that are not second-order neutralisable.

3.5. Feedback-invariant second-order conditions. In the previous section we saw that the second-order sufficient condition of Theorem 3.6 was not very sharp, as was illustrated via a pair of simple examples. Since the condition that a system Σ be second-order neutralisable is not feedback-invariant, one may try to better understand the condition by asking, "When is there a feedback transformation that transforms a system to one that is second-order neutralisable?" In this section we provide an answer to this question. The characterisation we give was essentially arrived at independently by Basto-Gonçalves [1998] and Hirschorn and Lewis [2001].

The conditions involve some ideas involving vector-valued quadratic forms. Thus let us develop these in generality for a moment. Let U and V be $\mathbb{R}$-vector spaces with $B\colon U\times U\to V$ a symmetric bilinear map. Given $\lambda\in V^*$ we define the symmetric bilinear function $B_\lambda\colon U\times U\to\mathbb{R}$ by

$$B_\lambda(u_1,u_2)=\langle\lambda;B(u_1,u_2)\rangle.$$

We may also define the function $Q_B\colon U\to V$ by $Q_B(u)=B(u,u)$. We also have $Q_{B_\lambda}(u)=B_\lambda(u,u)$ for $\lambda\in V^*$. Such a quadratic function as Q_{B_λ} has associated with it the usual notions of positive and negative-definiteness and semidefiniteness.[10] We shall say that B is ***definite*** if there exists $\lambda\in V^*$ so that Q_{B_λ} is positive-definite. We say that B is ***indefinite*** if for every $\lambda\in V^*$ the quadratic function Q_{B_λ} is not semidefinite. Let us be perfectly clear about this. Recall that for a given $\lambda\in V^*$, there exists a basis $\mathcal{B}=\{e_1,\dots,e_n\}$ for V so that the matrix $\mathrm{mat}_{\mathcal{B}}(B_\lambda)$ with components $B_\lambda(e_i,e_j)$, $i,j=1,\dots,n$, is diagonal with the diagonal entries taken from the set $\{0,1,-1\}$. Such a basis is called ***B_λ-orthonormal***. Then, B is definite when there exists $\lambda\in V^*$ so that all diagonal entries of $\mathrm{mat}_{\mathcal{B}}(B_\lambda)$ are $+1$ in a B_λ-orthonormal basis $\mathcal{B}$. B is indefinite if for every $\lambda\in V^*$, the nonzero diagonal entries of $\mathrm{mat}_{\mathcal{B}}(B_\lambda)$ do not all have the same sign in a B_λ-orthonormal basis.

Now let us define the vector-valued quadratic mapping of interest. We let $\Sigma=(M,\mathscr{F},U)$ and define

$$F_x=\mathrm{span}_{\mathbb{R}}(f_1(x),\dots,f_m(x))$$

so that F defines a distribution on M. A point $x\in M$ is a ***regular point*** for F if there is a neighbourhood $\mathcal{N}$ of x so that $\dim(F_y)=\dim(F_x)$ for every $y\in\mathcal{N}$. By $\pi_x\colon T_xM\to T_xM/F_x$ we denote the projection to the quotient vector space.[11] Let us fix $x\in M$ and assume that

[10] A quadratic function $Q\colon V\to\mathbb{R}$ is ***positive-definite*** if $Q(v)>0$ for $v\neq 0$ and is ***negative-definite*** if $-Q$ is positive-definite. Q is ***positive-semidefinite*** if $Q(v)\geq 0$ for all $v\in V$ and is ***negative-semidefinite*** if $-Q$ is positive-semidefinite.

[11] Let us recall what a quotient space is. Let V be a vector space with subspace U. A point in the quotient space V/U is a set of points of the form

$$\{v+u \mid u\in U\}$$

for some $v\in V$. We write such points in the suggestive manner $v+U$. The collection of all such objects

$f_0(x) = 0$ by the same reasoning as was used in the first-order case. We then define a T_xM/F_x-valued bilinear mapping on F_x by

$$\begin{aligned} B_\Sigma(x)\colon F_x \times F_x &\to T_xM/F_x \\ (u, v) &\mapsto \pi_x([U, [f_0, V]](x)), \end{aligned}$$

where U and V are vector fields having the property that $U(x) = u$ and $V(x) = v$. One needs to verify that this all makes sense, and that in particular the map is bilinear and does not depend on the way the vector fields U and V extend the tangent vectors u and v. However, everything does indeed make sense.

With this object at hand, we have the following result which gives a feedback-invariant characterisation of second-order neutralisability.

3.8 Theorem: *Let $\Sigma = (M, \mathscr{F} = \{f_0, f_1, \dots, f_m\}, U)$ be a control affine system with U proper and $f_0(x) = 0$ at some $x \in M$. The following statements are equivalent:*

(i) there exists $\tilde{m} \geq m$ and an injective linear map $L \in \mathrm{L}(\mathbb{R}^m; \mathbb{R}^{\tilde{m}})$ with the property that $\tilde{\Sigma} = (M, \tilde{\mathscr{F}} = \{f_0, \tilde{f}_1, \dots, \tilde{f}_{\tilde{m}}\}, \tilde{U})$ is second-order neutralisable with

(a) $\tilde{f}_\alpha = \sum_{a=1}^m L_{\alpha a} f_a$ and

(b) $\tilde{U} \subset \mathbb{R}^{\tilde{m}}$ proper;

(ii) $B_\Sigma(x)$ is indefinite.

Here are some comments.

3.9 Remarks: 1. The idea is that the condition that $B_\Sigma(x)$ be indefinite is the feedback-invariant answer of the question, "When is there a feedback transformation making Σ into a system that is second-order neutralisable?"

2. Clearly, one gets a sufficient condition for STLC at x by replacing condition (ii) in Theorem 3.6 with the condition that $B_\Sigma(x)$ be indefinite.

3. Here is a conjecture for necessity.

If x is a regular point for F and if $B_\Sigma(x)$ is definite, then Σ is not STLC at x.

Hirschorn and Lewis [2001] show that this is true for a certain class of mechanical systems.

4. The condition that $B_\Sigma(x)$ be indefinite or definite is one that can be checked using standard linear algebra.

Let's illustrate how to apply Theorem 3.8. We return to Examples 3.7–2 and 3.7–3. In particular, we show that in the first of these examples, where the hypotheses of Theorem 3.6 are not satisfied, that $B_\Sigma(0, 0, 0)$ is indefinite.

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forms a vector space with vector addition and scalar multiplication defined by

$$(v_1 + U) + (v_2 + U) = (v_1 + v_2) + U, \quad \alpha(v + U) = (\alpha v) + U.$$

Intuitively, one should think of V/U as representing a complement to U in V. Indeed, if W is any complement of U in V, then there exists a *natural* isomorphism from W to V/U.

3.10 Examples: Before we present the results of the calculations, let us say how we got them. In each case the tangent space is three-dimensional and the input distribution is two-dimensional. Thus the quotient $T_xM/C(\mathscr{F})_x^{(1)}$ is one-dimensional. Therefore, in this case, $B_\Sigma(x)$ is essentially a regular symmetric bilinear form. In the following examples, we simply write the matrix for $B_\Sigma(x)$ in an "obvious" basis. That is, we use the input vector fields as the basis for F_x, and the subspace $\mathrm{span}_{\mathbb{R}}((1,0,0))$ as a model for $T_xM/C(\mathscr{F})_x^{(1)}$.

1. Let us consider the input vectors fields $\{f_1, f_2\}$ as given in Example 3.7–2. Here we have
$$B_\Sigma(0,0,0) = \begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix}.$$
Note that second-order neutralisability essentially amounts, in this case, to the sum of the diagonals being zero. This is not the case here. However, one may verify that $B_\Sigma(0,0,0)$ is indefinite. In this case this simply amounts to the determinant of $B_\Sigma(0,0,0)$ being negative.
2. Now we consider the input vector fields $\{\tilde{f}_1, \tilde{f}_2\}$ given in Example 3.7–3. In this case we have
$$B_\Sigma(0,0,0) = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$
In this case, second-order neutralisability is reflected by the fact that the sum of the diagonals is zero. And again, indefiniteness may be verified here by checking that the determinant is negative. •

4. Open questions

Here are some more or less obvious open questions suggested by the above developments concerning controllability.

1. Is the conjecture of Remark 3.9–3 true? To prove this one must, it appears, understand well some series expansion results for control systems. The technology presented by Agrachev and Gamkrelidze [1978] and/or by Sussmann [1983a] is promising, perhaps. The necessary condition of Hirschorn and Lewis [2001] relies on a series expansion of Bullo [1999], which in turns rests on the results of Agrachev and Gamkrelidze [1978].
2. In the program outlined above, one of the issues will be determining when a given set of conditions, typically a necessary and a sufficient condition of some order, are "sharp." In the zeroth and first-order cases, this was accomplished by the notions of STLC_0 and STLC_1. Attendant to these were notions of zeroth and first-order equivalence of control affine systems. The notions used by Sussmann [1978] in the zeroth-order case, and by Bianchini and Stefani [1984] in the first-order case, provide equivalence in terms of comparing the *exact* values for the systems at the point of interest. This seems a very stringent notion of equivalence. What's more, it reacts poorly with feedback-invariance. What are the proper notions of pointwise kth-order equivalence?
3. In Section 3.5 a slick geometric/algebraic construct provides what seems to be *the* relevant object in discussing second-order obstructions to controllability. This object is distinguished by its feedback-invariance. For higher-order obstructions, there are Lie bracket characterisations much like that for second-order neutralisability (see [Sussmann

1987]). Are there slick geometric/algebraic characterisations for these obstructions that are feedback-invariant?

4. Can one understand control design issues by better understanding local controllability? For example, can one devise systematic local stabilisation algorithms, or trajectory planning algorithms using the tools for local controllability. This approach works for certain classes of systems. For example, Morin, Pomet, and Samson [1999] give a general stabilisation methodology for systems without drift.

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