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Geometric Methods for Nonlinear Optimal Control Problems[1]

J. BAILLIEUL[2]

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Abstract. It is the purpose of this paper to develop and present new approaches to optimal control problems for which the state evolution equation is nonlinear. For bilinear systems in which the evolution equation is right invariant, it is possible to use ideas from differential geometry and Lie theory to obtain explicit closed-form solutions.

Key Words. Nonlinear systems, calculus of variations, necessary conditions, maximum principle, optimization theorems.

1. Introduction

The optimization theory for dynamical systems with linear state evolution equation and quadratic performance measure is well understood. Results for optimization problems involving nonlinear evolution equations, however, are not nearly so extensive. Traditional approaches to such problems have involved linearization and numerical techniques. It is the purpose of this paper to examine ways in which the structure intrinsic to a nonlinear problem may be used to find a solution. For right-invariant systems evolving on matrix Lie groups, many optimization problems admit closed-form analytic solutions of simplicity comparable to the linear–quadratic case.

There has been relatively little reported in the literature concerning explicit determination of optimal controls for nonlinear problems. Notable exceptions include work by Brockett (Ref. 1), Baillieul (Refs. 2 and 3), and Jacobson (Ref. 4). The work of Brockett and Baillieul is extended in the present paper. In proving Theorem 3.2, we apply the maximum principle in

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[2] Assistant Professor of Mathematics, Georgetown University, Washington, DC.

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much the same way that Brockett has, but our results are stronger in that we go on to establish the *normality* (see Section 3) of certain classes of problems. Brockett also considers the general fixed endpoint problem on $SO(n)$ involving the state evolution equation

$$(d/dt)X(t) = \Omega(t)X(t)$$

and performance measure

$$\eta = -\int_0^T \operatorname{tr}(K^{-1}\Omega)^2\, dt,$$

where K is a matrix of weights. To solve this problem, a variational argument is used to show that the optimal $\Omega(t)$ satisfies

$$(d/dt)\Omega(t) = [\Omega K \Omega, K^{-1}].$$

We have generalized this to a wider class of systems, performance measures, and state spaces.

In Section 2, we assume our performance measure is of the form

$$\eta = \int_0^T L(x, \dot{x})\, dt.$$

If $L_{\dot{x}\dot{x}}$ (or equivalently $L_{\dot{x}\dot{x}}{}^{-1}$) is positive definite, this is called a *regular problem* in the calculus of variations, and necessary conditions for regular problems are obtained in Theorem 2.1. Theorem 2.2 shows that certain constants of motion exist for regular problems, and Example 2.1 shows that some classical results in rigid body dynamics may be deduced from this. Later, we use penalty function techniques and the maximum principle to treat the more general case where $L_{\dot{x}\dot{x}}{}^{-1}$ is only assumed to be nonnegative definite, i.e., $L_{\dot{x}\dot{x}}$ has become infinite in certain *directions*. The final section of this paper is devoted to methods for obtaining explicit closed-form functions.

It is assumed that the reader is familiar with the elementary aspects of the theory of Lie groups and Lie algebras (Refs. 5–7). We shall briefly review the role that this theory plays in the analysis of systems of the form

$$(d/dt)X(t) = \left(A + \sum_{i=1}^{m} u_i(t)B_i\right)X(t), \tag{1}$$

where $A, B_1, \ldots, B_m$ are constant $n \times n$ matrices, $X(t)$ is a time-varying $n \times n$ matrix, and the $u_i(\cdot)$'s are functions which belong to some admissible class Ω, such as measureable functions on $[0, T]$. Systems of this type are called *right invariant*, since, if any solution $X(t)$ is multiplied on the right by a constant matrix M, the product $X(t) \cdot M$ also satisfies (1).

Let $\{A, B_1, \ldots, B_m\}_{LA}$ denote the Lie algebra generated by the matrices $A, B_1, \ldots, B_m$; and, for any Lie algebra $\mathfrak{g}$, let $\{\exp \mathfrak{g}\}_G$ denote the group obtained by taking all finite products of exponentials of elements of $\mathfrak{g}$. See Ref. 8 for the basic facts concerning these constructions. It is well known that, if

$$X(0) = X_0$$

is an element of

$$G = \{\exp\{A, B_1, \ldots, B_m\}_{LA}\}_G,$$

then $X(t)$, for any admissible controls $u_i(t)$, $i = 1, \ldots, m$, will lie in G for all $t \geq 0$.

Optimization Problem. Suppose that G is a matrix Lie group with corresponding matrix Lie algebra $\mathfrak{g}$. Consider the system defined on G by (1), where $A, B_1, \ldots, B_m \in \mathfrak{g}$. Given an admissible class Ω of control functions, we wish to find $u_1, \ldots, u_m$ in Ω which steer the state of (1) from $I \in G$ (the identity) to $X_1 \in G$ in T units of time in such a way as to minimize the cost functional

$$\eta = \int_0^T \sum_{i,j=1}^{m} q_{ij} u_i(t) u_j(t)\, dt, \tag{2}$$

where

$$Q = (q_{ij})$$

is a symmetric, positive-definite matrix.

Remark 1.1. An alternative statement of this problem might replace the initial condition

$$X(0) = I$$

with

$$X(0) = X_0 \in G.$$

However, because (1) is right invariant, no generality is gained by doing this.

Remark 1.2. It will be assumed throughout this paper that the matrices B_i appearing in (1) are linearly independent. In addition, it will be convenient to assume they are orthonormal with respect to the usual matrix inner product

$$\langle X, Y \rangle = \operatorname{tr}(X \cdot {}^tY).$$

Here, ${}^t Y$ denotes the transpose of Y. In this case, we may define a linear transformation

$$\tilde{Q} : h_0 \to h_0$$

(h_0 is the linear span of the B_i's) by

$$\tilde{Q}(B_j) = \sum_{i=1}^{m} q_{ij} B_i.$$

Then, (2) may be rendered as

$$\eta = \int_0^T \left\langle \sum u_i(t) B_i, \tilde{Q}\left(\sum u_i(t) B_i\right) \right\rangle dt.$$

Remark 1.3. Although we state all our results in terms of subgroups of $GL(n, \mathbb{R})$, they are easily extended to the real forms of complex groups. Indeed, if

$$\mathfrak{g} \subseteq \mathfrak{gl}(n, \mathbb{R}),$$

define the realification of $\mathfrak{g}$ as the image of the Lie algebra homomorphism

$$B \to \begin{bmatrix} \operatorname{Re} B & \operatorname{Im} B \\ -\operatorname{Im} B & \operatorname{Re} B \end{bmatrix}.$$

If the matrices $A, B_1, \ldots, B_m$ have complex entries, we may rewrite (1) in terms of the above realification mapping (see Example 4.1).

Remark 1.4. Since the goal of this paper is the development of necessary conditions, the existence of optimal controls must be proved. Standard results in existence theory (see, e.g., Ref. 6, p. 411) cover all cases treated in this paper.

2. Calculus of Variations

Specializing the results of the calculus of variations to problems defined on Lie groups is relatively straightforward and has been carried out along slightly different lines by Hermann (Ref. 10) and Brockett (Ref. 1); the latter work has been detailed in Ref. 2. We shall draw on the results of Brockett and Baillieul to solve our optimization problem. In order to use the calculus of variations, we assume that the B_i in (1) span $\mathfrak{g}$. In the light of the assumptions set out in Section 1, this makes $\{B_1, \ldots, B_m\}$ a basis for $\mathfrak{g}$.

Theorem 2.1. Consider the system (1) under the above hypothesis, and let R be a nonsingular matrix either symmetric or skew-symmetric such that

$$R^2 = \pm I.$$

Suppose that

$$\mathfrak{g} = \{C \in \mathfrak{gl}(n, \mathbb{R}) : {}^tCR + RC = 0\}.$$

Let

$$X_1 \in G = \{\exp \mathfrak{g}\}_G$$

and $T > 0$ be given. Then, the following results hold:

(i) there exists an optimal control matrix

$$U^0(t) = \sum_{i=1}^{m} u_i^0(t) B_i$$

which steers (1) from I at $t = 0$ to X_1 at $t = T$ such that (2) is minimized;

(ii) the optimal control matrix

$$U^0(t) = \sum_{i=1}^{m} u_i^0(t) B_i$$

satisfies the differential equation

$$(d/dt)\left[\sum_{i=1}^{m} u_i(t)B_i\right] = \tilde{Q}^{-1}\left(\left[\tilde{Q}\left(\sum u_i(t)B_i\right), {}^tA + \sum u_i(t){}^tB_i\right]\right), \tag{3}$$

where $[\cdot\,,\cdot]$ denotes the Lie bracket of matrices

$$[X, Y] = XY - YX.$$

Proof. (i) follows from standard results on the controllability of bilinear systems (Ref. 8) and the results on existence of optimal controls cited in the introduction.

From Theorem 1 in Ref. 2, we know that the optimal trajectory must satisfy the differential equation

$$P\{(\nabla_x L - (d/dt)\nabla_{\dot{x}} L)^t X\} = 0, \tag{4}$$

where

$$L = \langle \dot{X}X^{-1} - A, \tilde{Q}(\dot{X}X^{-1} - A)\rangle,$$

$\nabla_x L$ and $(d/dt)\nabla_{\dot{x}} L$ are vector fields on $GL(n, \mathbb{R})$ given in local coordinates by

$$\nabla_x L = \partial L/\partial x_{ij}, \qquad (d/dt)\nabla_{\dot{x}} L = (d/dt)(\partial L/\partial \dot{x}_{ij}),$$

and P is the orthogonal projection of $\mathfrak{gl}(n, \mathbb{R})$ onto $\mathfrak{g}$. For the present case, it is easy to check that

$$\nabla_x L = 2\tilde{Q}\left(\sum u_i(t)B_i\right)\left({}^tA + \sum u_i(t){}^tB_i\right)RX(t){}^tR,$$

$$(d/dt)\nabla_x L = 2\left\{\tilde{Q}\left(\sum (du_i/dt)B_i\right) - \tilde{Q}\left(\sum u_i(t)B_i\right)\left({}^tA + \sum u_i(t)B_i\right)\right\}{}^tRX(t)R,$$

and

$$P(C) = \tfrac{1}{2}(C - R{}^tC{}^tR),$$

whence the theorem follows.

We shall call Eq. (3) the *Euler–Lagrange equation* for our optimization problem.

Theorem 2.2. Suppose that the collection of matrices $A, B_1, \ldots, B_m$ is self-contragredient, i.e.,

$$A = -{}^tA, \qquad B_i = -{}^tB_i.$$

If

$$U^0(t) = \sum u_i^0(t)B_i$$

is an optimal control, then

$$\langle A + U^0(t), \tilde{Q}(A + U^0(t))\rangle \equiv K_1, \tag{5}$$

$$\langle \tilde{Q}(U^0(t)), \tilde{Q}(U^0(t))\rangle \equiv K_2, \tag{6}$$

where K_i is a constant positive real number independent of t.

Proof. If $A, \ldots, B_m$ are self-contragredient, then (4) may be rewritten as

$$(d/dt)\tilde{Q}\left(\sum u_i^0(t)B_i\right) = \left[A + \sum u_i^0(t)B_i, \tilde{Q}\left(\sum u_i^0(t)B_i\right)\right].$$

Hence,

$$\begin{aligned}(d/dt)&\left\langle A + \sum u_i^0(t)B_i, \tilde{Q}\left(A + \sum u_i^0(t)B_i\right)\right\rangle \\ &= 2\left\langle\left[A + \sum u_i^0(t)B_i\right], \left[A + \sum u_i^0(t)B_i, \tilde{Q}\left(\sum u_i^0(t)B_i\right)\right]\right\rangle \\ &= 2\left\langle\left[A + \sum u_i^0(t)B_i, A + \sum u_i^0(t)B_i\right], \tilde{Q}\left(\sum u_i^0(t)B_i\right)\right\rangle,\end{aligned}$$

which follows since

$$\langle X, [Y, Z]\rangle = \langle [X, Y], Z\rangle \qquad \text{for all } X, Y, Z \in \mathfrak{g}$$

if $\mathfrak{g}$ is self-contragredient. This last expression is identically zero. Expression (6) is proved in the same way.

Viewing $\mathfrak{g}$ as a finite-dimensional vector space and writing (5) and (6) out in terms of coordinates, Theorem 2.2 says that the optimal controls for our problem lie on the intersection of two quadratic hypersurfaces in $\mathfrak{g}$.

Example 2.1. Theorems 2.1 and 2.2 are of special interest in the case that

$$G = SO(3).$$

Consider the system

$$(d/dt)X(t) = \begin{bmatrix} 0 & u_3(t) & -u_2(t) \\ -u_3(t) & 0 & u_1(t) \\ u_2(t) & -u_1(t) & 0 \end{bmatrix} X(t)$$

and the performance criterion

$$\eta = \int_0^T [q_1 u_1(t)^2 + q_2 u_2(t)^2 + q_3 u_3(t)^2]\, dt, \qquad q_i > 0.$$

From Theorem 2.1, we find that optimal controls steering this system between fixed endpoints satisfy

$$\begin{aligned} q_1(du_1/dt) &= (q_2 - q_3)u_2 u_3, \\ q_2(du_2/dt) &= (q_3 - q_1)u_1 u_3, \\ q_3(du_3/dt) &= (q_1 - q_2)u_1 u_2. \end{aligned} \tag{7}$$

Interpreting the u_i's as angular velocities and the q_i's as moments of inertia about the principal axes, the optimization problem corresponds to the problem in classical mechanics of finding the equations of motion of a rotating rigid body in the absence of external torques. Here, η is the action, (7) are Euler equations, and (5)–(6) show that kinetic energy and magnitude of angular momentum are conserved.

Unfortunately, if we do not assume that the B_i's span the Lie algebra $\mathfrak{g}$, then (3) is no longer a necessary condition for the optimal control. Nevertheless, techniques exist which allow us to develop the requisite necessary condition, even when the B_i's do not span. One approach is the maximum principle of the next section. Alternatively, we can use a limiting argument coupled with (3). Associated to the optimization problem of Section 1, define the following auxiliary problems.

λ-Auxiliary Problems. Let $G, \mathfrak{g}$ be as in the previous sections, and let $h_0 \subseteq \mathfrak{g}$ be the linear span of the B_i's appearing in (1). In general, h_0 is a proper subspace of $\mathfrak{g}$. Consider the system

$$(d/dt)X(t) = (A + U(t))X(t), \tag{8}$$

where A, $U(t) \in \mathfrak{g}$ and for which we have defined the cost functional

$$\eta_\lambda = \int_0^T \langle U(t), \tilde{Q}_\lambda(U(t))\rangle \, dt,$$

where

$$\tilde{Q}_\lambda : \mathfrak{g} \to \mathfrak{g}$$

is the linear mapping defined on $\mathfrak{g}$ by

$$\tilde{Q}_\lambda(H) = \begin{cases} \lambda H & \text{if } H \in h_1 = h_0^\perp, \\ \tilde{Q}(H) & \text{if } H \in h_0, \end{cases}$$

where $\tilde{Q}$ is the analogous linear mapping

$$\tilde{Q} : h_0 \to h_0$$

associated with the performance measure for our primary optimization problem. It is desired to find an $L_2[0, T]$ control matrix $U_\lambda(t)$ taking values in $\mathfrak{g}$ which steers (8) from

$$I \in G \qquad \text{at } t = 0$$

to

$$X_1 \in G \qquad \text{at } t = T,$$

so as to minimize η_λ. We shall assume for the purpose of the next theorem that all Lie groups and algebras under discussion are self-contragredient, i.e.,

$$C = -{}^tC \qquad \text{for all } C \in \mathfrak{g}$$

and

$$X = {}^tX^{-1} \qquad \text{for all } X \in G.$$

Theorem 2.3. Together with this assumption on G and $\mathfrak{g}$, assume that (1) is controllable on G, in the sense that, given $T > 0$ and any

$X_0, X_1 \in G$, there exists a control matrix $U(t)$ taking values in h_0 whose entries are $L_2[0, T]$ and which steers (1) from X_0 at time $t=0$ to X_1 time $t=T$. Then, the following results hold:

(i) there is at least one control

$$U^0(t)=\sum_{i=1}^{m} u_i^0(t)B_i$$

which solves the primary problem as stated on page 521;

(ii) there is a sequence $\lambda_j \to \infty$ as $j\to\infty$ such that the solutions U^{λ_j} to the λ_j-auxiliary problems converge a.e. on $[0, T]$ to an optimal control solving the primary problem and such that the corresponding trajectories X_{λ_j} converge uniformly on $[0, T]$ to an optimal trajectory; moreover, if U^0 is unique, then, for *any* such sequence of λ_j, the U^{λ_j} converge a.e. to U^0 and the X_{λ_j} converge uniformly to the corresponding optimal trajectory.

The proof of this theorem is too lengthy to be included here, but may be found in Ref. 3.

3. Maximum Principle

A more direct approach to the optimization problem of Section 1 is to invoke the high-order maximum principle developed by Krener (see Ref. 11). The statement of the high-order maximum principle requires the definition of a control variation. Let $X(t, U(\cdot), Z)$ denote the trajectory generated via (1) by the control

$$U(\cdot)=\sum_{i=1}^{m} u_i(\cdot)B_i$$

and initiating at the point Z at time $t=0$. Let us assume in this section that admissible controls $u_i(\cdot)$ are piecewise C^∞ on $[0, T]$.

Definition 3.1. (i) Consider the trajectory $X(t, U^0(\cdot), X_0)$ generated by $U^0(\cdot)$ and initiating at X_0. A *control variation* to the control $U^0(\cdot)$ at $X(t_1)$ is given by the pair (Φ_s, ϕ_s), where

$$\Phi_s(X(t_1), U^0)=X(p_k(s), U^k(\cdot), X(p_{k-1}(s), U^{k-1}(\cdot), \ldots,$$
$$X(p_1(s), U^1(\cdot), X(t_1-\sum_{i=1}^{k} p_i(s), U^0(\cdot), X_0))\ldots),$$

and

$$\begin{aligned}\phi_s(U^0(t_1)) = &\int_{t_1-p_k(s)}^{t_1} \langle U^k(\sigma), \tilde{Q}(U^k(\sigma))\rangle\, d\sigma \\ &+ \int_{t_1-p_k(s)-p_{k-1}(s)}^{t_1-p_k(s)} \langle U^{k-1}(\sigma), \tilde{Q}(U^{k-1}(\sigma))\rangle\, d\sigma \\ &+\cdots+ \int_{t_1-p_k(s)-\cdots-p_1(s)}^{t_1-p_k(s)-\cdots-p_2(s)} \langle U^1(\sigma), \tilde{Q}(U^1(\sigma))\rangle\, d\sigma \\ &+ \int_0^{t_1-p_k(s)-\cdots-p_1(s)} \langle U^0(\sigma), \tilde{Q}(U^0(\sigma))\rangle\, d\sigma.\end{aligned}$$

The quantities $U^0(\cdot)$, $U^1(\cdot), \ldots, U^k(\cdot)$ are control matrices

$$U^j(\cdot) = \sum_{i=1}^{m} u_i^j(\cdot) B_i,$$

and the $p_i(\cdot)$ are polynomials in s satisfying

$$p_i(0) = 0$$

and

$$p_i(s) \geq 0 \qquad \text{for small } s \geq 0.$$

(ii) A control variation is said to be of *order* h at $X(t_1)$ if there exists $\epsilon > 0$ such that

$$(d^j/ds^j)_{s=0}\Phi_s(X(t), U^0(t)) = 0, \qquad (d^j/ds^j)_{s=0}\phi_s(U^0(t)) = 0,$$

for

$$j = 1, \ldots, h-1 \qquad \text{and} \qquad |t - t_1| < \epsilon.$$

For the optimization problem presented in Section 1, the high-order maximum principle asserts that, if

$$U^0(\cdot) = \sum_{i=1}^{m} u_i^0(\cdot) B_i$$

is an optimal control and $X_0(\cdot)$ is the corresponding trajectory satisfying (1), there is a constant $\psi_0 \geq 0$ and a matrix $\Psi(t)$ satisfying

$$(d/dt)\Psi(t) = -({}^t\!A + u_i^0(t)\,{}^t\!B_i)\Psi(t), \tag{9}$$

such that the Hamiltonian

$$H(\Psi, \psi_0, X_0, U) = \langle \Psi(t), \Big(A + \sum u_i B_i\Big) X_0(t)\rangle + \tfrac{1}{2}\psi_0 \sum q_{ij} u_i u_j \tag{10}$$

is minimized with respect to U by

$$U^0(t) = \sum u_i^0(t) B_i.$$

Moreover, for every control variation (Φ_s, ϕ_s) to $U^0(\cdot)$ at $X_0(t)$ which is of order h,

$$\langle \Psi(t), (d^h/ds^h)_{s=0} \Phi_s(X_0(t), U^0(t)) \rangle + \tfrac{1}{2}\psi_0 (d^h/ds^h)_{s=0} \phi_s(U^0(t)) \geq 0; \quad (11)$$

finally, if

$$\psi_0 = 0,$$

then

$$P(\Psi(t)^t X_0(t)) \neq 0 \qquad \text{for any } t \text{ in } [0, T].$$

Recall that

$$P : \mathfrak{g}l(n, R) \to \mathfrak{g}$$

is the orthogonal projection with respect to our matrix inner product $\langle \cdot, \cdot \rangle$.

Remark 3.1. If U is any point in h_0 (the linear span of the B_i's), then, by using the variation

$$\Phi_s(X_0(t), U^0(t)) = X(s, U, X(t-s, U^0(\cdot), I)),$$

the fact that the Hamiltonian is minimized by $U^0(\cdot)$ is seen to follow as a special case of (11).

The maximum principle specialized to optimization problems on matrix Lie groups leads to the following result.

Theorem 3.1. Suppose that (1) evolves on a matrix Lie group with corresponding Lie algebra $\mathfrak{g}$. Let

$$U^0(\cdot) = \sum_{i=1}^m u_i^0(\cdot) B_i$$

be an optimal control and $X_0(\cdot)$ the optimal trajectory corresponding to $U^0(\cdot)$ via (1). Then, there exists a constant $M \in \mathfrak{g}$ and a nonnegative real number ψ_0, not both zero, such that

$$H(M, \psi_0, U, X_0(t)) = \left\langle M, X_0(t)^{-1}\left(A + \sum u_i B_i\right) X_0(t) \right\rangle + \tfrac{1}{2}\psi_0 \sum q_{ij} u_i u_j$$

is minimized with respect to

$$U = \sum u_i B_i$$

by $U^0(t)$. Moreover, for every control variation (Φ_s, ϕ_s) to $U^0(\cdot)$ at $X_0(t)$ which is of order h,

$$\langle M, X_0(t)^{-1}(d^h/ds^h)_{s=0}\Phi_s(X_0(t), U^0(t))\rangle + \tfrac{1}{2}\psi_0(d^h/ds^h)_{s=0}\phi_s(U^0(t)) \geq 0. \tag{12}$$

Proof. From (1), it follows that

$$(d/dt)^t X_0(t)^{-1} = -\left[{}^tA + \sum_{i=1}^m u_i^0(t)^t B_i\right]{}^t X_0(t)^{-1}.$$

Since

$${}^t X_0(0)^{-1} = I,$$

there exists some constant matrix M such that

$$\Psi(t) = {}^t X_0(t)^{-1} M.$$

The Hamiltonian in (10) may thus be rewritten as

$$\left\langle {}^t X_0(t)^{-1}M, \left(A + \sum u_i B_i\right) X_0(t)\right\rangle + \tfrac{1}{2}\psi_0 \sum q_{ij}u_i u_j,$$

which may further be rewritten as

$$\left\langle M, X_0(t)^{-1}\left(A + \sum u_i B_i\right) X_0(t)\right\rangle + \tfrac{1}{2}\psi_0 \sum q_{ij}u_i u_j.$$

The theorem follows directly from the maximum principle, once we establish the claim that $M \in \mathfrak{g}$. But this is evidently the case, since the value of H can in no way be influenced by any component of M lying in $\mathfrak{g}^\perp$.

Definition 3.2. Problems for which it is assured that

$$\psi_0 \neq 0$$

are called *normal*.

It is clear that, for normal problems, we may normalize ψ_0 to be 1, and optimal controls may be found by differentiating H with respect to u_i for $i = 1, 2, \ldots, m$ and setting the result equal to zero. From the preceding theorem, we obtain

$$u_i^0(t) = -\sum_{j=1}^m q^{ij}\langle M, X_0(t)^{-1}B_j X_0(t)\rangle,$$

where

$$(q^{ij}) = Q^{-1}.$$

Unfortunately, not all of our optimization problems are normal, as may be seen from the following example where the state space is $SO(3)$.

Example 3.1. It is desired to steer the system

$$(d/dt)X(t)=\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & u(t) \\ 0 & -u(t) & 0 \end{bmatrix} X(t)$$

from

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{at time } t=0$$

to

$$X_1=\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{at time } t=\pi/2,$$

so as to minimize

$$\eta=\int_0^{\pi/2} u(t)^2\,dt.$$

The optimal control is obviously given by

$$u(t)\equiv 0,$$

and we let $X_0(t)$ be the corresponding trajectory. By Theorem 3.1, there is a constant matrix

$$M=\begin{bmatrix} 0 & m_3 & -m_2 \\ -m_3 & 0 & m_1 \\ m_2 & -m_1 & 0 \end{bmatrix}$$

and real

$$\psi_0 \geqslant 0,$$

not both zero, such that

$$H(M, \psi_0, u, X_0(t))=\left\langle M, X_0(t)^{-1}\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & u \\ 0 & -u & 0 \end{bmatrix} X_0(t)\right\rangle+\tfrac{1}{2}\psi_0 u^2$$

is minimized with respect to u by the optimal control. Taking

$$m_1 = m_2 = \psi_0 = 0, \qquad m_3 = \alpha \neq 0,$$

we have

$$H \equiv 2\alpha,$$

and we see this problem is not normal.

To this author's knowledge, there is no exact characterization of normality, although it is discussed at some length in Ref. 12. The following theorems present some useful sufficient conditions.

Theorem 3.2. Consider the system (1) evolving on a matrix Lie group G with corresponding Lie algebra $\mathfrak{g}$. If $X_0(t)$ is the optimal trajectory for the problem given in Section 1, and if $X_0(t)^{-1}B_iX_0(t)$ does not vanish identically on any subspace of $\mathfrak{g}$ for all $i = 1, \ldots, m$, then the problem is normal.

Proof. Suppose that the problem is not normal, i.e.,

$$\psi_0 = 0.$$

Then, differentiating the Hamiltonian with respect to u_i and setting the result equal to zero yields

$$\langle M, X_0(t)^{-1}B_iX_0(t)\rangle \equiv 0, \qquad i = 1, \ldots, m.$$

Since

$$\psi_0 = 0$$

implies that

$$M \neq 0,$$

the above shows that $X_0(t)^{-1}B_iX_0(t)$ vanishes on the ray in $\mathfrak{g}$ spanned by M, in violation of the hypothesis of the theorem.

Example 3.2. Suppose that in (1) the matrices $A, B_1, \ldots, B_m$ are all $(n+1)\times(n+1)$ and

$$A = \begin{bmatrix} \tilde{A} & 0 \\ 0 & 0 \end{bmatrix},$$

where $\tilde{A}$ is an $n \times n$ constant matrix,

$$B_i = \begin{bmatrix} 0 & b_i \\ 0 & 0 \end{bmatrix},$$

where b_i is an n-dimensional column vector, so that the system evolves on Aff(n), the group of affine transformations of R^n. By making the time-varying change of coordinates

$$Z(t) = \exp(-At)X(t),$$

we find that (1) in fact evolves on the subgroup

$$G_0 = \left\{ \left[\begin{array}{cccc|c} 1 & 0 & \dots & 0 & x_1 \\ 0 & 1 & \dots & 0 & x_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 1 & x_n \\ \hline 0 & 0 & \dots & 0 & 1 \end{array}\right] : x_i \in \mathbb{R} \right\},$$

which is the image of the canonical embedding of $\mathbb{R}^n$ into Aff(n). The corresponding subalgebra is

$$\mathfrak{g}_0 = \left\{ \left[\begin{array}{cccc|c} 0 & 0 & \dots & 0 & y_1 \\ 0 & 0 & \dots & 0 & y_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 0 & y_n \\ \hline 0 & 0 & \dots & 0 & 0 \end{array}\right] : y_i \in \mathbb{R} \right\}.$$

Equation (1) thus corresponds to the evolution equation of the system

$$dx/dt = \tilde{A}x + Bu,$$

where B is the $n \times n$ matrix whose ith column is b_i. It is well known that our optimization problem will be normal if the corresponding linear system is controllable. This may be seen using Theorem 3.2 as follows. If $X_0(t)^{-1}B_iX_0(t)$ were to vanish identically on any subspace of the subalgebra $\mathfrak{g}_0$, then so would all derivatives of $X_0(t)^{-1}B_iX_0(t)$. It is a straightforward calculation to show that

$$\begin{aligned}(d^k/dt^k)X_0(t)^{-1}B_iX_0(t) &= (-1)^k X_0(t)^{-1}[\mathrm{ad}^k_{A+\sum u_i^0(t)B_i}(B_i)]X_0(t) \\ &= \begin{bmatrix} 0 & (-1)^k \exp(-\tilde{A}t)\tilde{A}^k b_i \\ 0 & 0 \end{bmatrix},\end{aligned}$$

where we recall that, for any $C, D \in \mathfrak{g}$,

$$\mathrm{ad}_C^0(D) = D, \qquad \mathrm{ad}_C^1(D) = [C, D],$$
$$\mathrm{ad}_C^k(D) = [C, \mathrm{ad}_C^{k-1}(D)] \qquad \text{for } k \geq 2,$$

a positive integer. If, for $i = 1, \ldots, m$, these derivatives vanish for $k = 0, 1, 2, \ldots$ on any subspace of $\mathfrak{g}_0$, then $(B, \tilde{A}B, \ldots, \tilde{A}^{n-1}B)$ cannot have rank n. Hence, in this case, the standard controllability condition for linear systems would not be met, and therefore controllability implies normality.

The following sufficient condition for normality will prove useful in the next section.

Theorem 3.3. (Ref. 13). Suppose that (1) evolves on a matrix Lie group G with corresponding Lie algebra $\mathfrak{g}$. Let h_0 be the linear subspace of $\mathfrak{g}$ spanned by $B_1, \ldots, B_m$, and suppose that

$$\mathfrak{g} \subseteq h_0 + [h_0, h_0],$$

i.e., any element in $\mathfrak{g}$ may be written as a linear combination of the B_i's and the first-order brackets $[B_i, B_j]$. For such a system, the optimization problem set out in Section 1 is normal.

Proof. Assume, contrary to the conclusion of the theorem, that

$$\psi_0 = 0.$$

Then, the Hamiltonian defined in Theorem 3.1 is

$$H(M, U, X_0(t)) = \left\langle M, X_0(t)^{-1}\left(A + \sum u_i B_i\right) X_0(t)\right\rangle.$$

Since this is to be minimized with respect to U, we set the partial derivatives

$$\partial H/\partial u_i = \langle M, X_0(t)^{-1} B_i X_0(t)\rangle \equiv 0, \qquad i = 1, \ldots, m. \tag{13}$$

Also from Theorem 3.1, we know that, for any control variation (Φ, ϕ) of order h to the optimal control,

$$\left\langle M, X_0(t)^{-1}(d^h/ds^h)_{s=0}\Phi_s(X_0(t), U^0(t))\right\rangle \geq 0.$$

Note that, since $\psi_0 = 0$, ϕ does not enter. In particular, let $t_1 \in (0, T)$, and consider

$$\Phi_s(X_0(t_1), U^0(t_1))$$
$$= X(s, V(\cdot), X(s, U^i(\cdot), X(s, U^j(\cdot), X(s, V(\cdot), X_0(t_1 - 4s))))),$$

where $U^i(\cdot)$ is the constant control matrix B_i, $U^j(\cdot)$ is the constant control matrix B_j, and $V(\cdot)$ is the control matrix

$$V(t)=2U^0(t_1-4t)-\tfrac{1}{2}B_i-\tfrac{1}{2}B_j.$$

To see that this is a second-order control variation, define

$$\Phi(s_0, s_1, s_2, s_3, s_4) = X(s_1, V(\cdot), X(s_2, U^i(\cdot), X(s_3, U^j(\cdot), X(s_4, V(\cdot), X_0 s_0)))),$$

and let

$$s_i = s_i(s) = s, \qquad i=1,2,3,4,$$

while

$$s_0 = s_0(s) = t_1 - 4s.$$

Then,

$$(d\Phi_s/ds)_{s=0} = \sum_{i=0}^{4} ((\partial\Phi/\partial s_i)\cdot(ds_i/ds))_{s=0} = 0.$$

Also,

$$\begin{aligned}(d^2\Phi_s/ds^2)_{s=0} &= \sum_{i,j=0}^{4} ((\partial^2\Phi/\partial s_i\partial s_j)\cdot(ds_i/ds)\cdot(ds_j/ds))_{s=0}\\ &= \sum_{i=0}^{4} (\partial^2\Phi/\partial s_i^2)\cdot[(ds_i/ds)_{s=0}]^2\\ &\quad +2\sum_{0\leqslant i\leqslant j\leqslant 4} ((\partial^2\Phi/\partial s_i\partial s_j)\cdot(ds_i/ds)\cdot(ds_j/ds))_{s=0}\\ &= [A+B_i, A+B_j]X_0(t_1).\end{aligned}$$

Next, consider the variation

$$\tilde{\Phi}_s(X_0(t_1), U^0(t_1)) = X(s, W(\cdot), X(s, -U^i(\cdot), X(s, -U^j(\cdot), X(s, W(\cdot), X_0(t_1-4s))))),$$

where U^i and U^j are as above and $W(\cdot)$ is the control matrix

$$W(t)=2U^0(t_1-4t)+\tfrac{1}{2}B_i+\tfrac{1}{2}B_j.$$

Repeating the above calculations, we find this is also a second-order variation and

$$(d^2\tilde{\Phi}_s/ds^2)_{s=0} = [A-B_i, A-B_j]X_0(t_1).$$

Now, since

$$\langle M, X_0(t_1)^{-1}[A+B_i, A+B_j]X_0(t_1)\rangle \geq 0,$$
$$\langle M, X_0(t_1)^{-1}[A-B_i, A-B_j]X_0(t_1)\rangle \geq 0,$$

by adding these inequalities, we find that

$$\langle M, X_0(t_1)^{-1}[B_i, B_j]X_0(t_1)\rangle \geq 0.$$

But it is clear that, by reversing the order of the controls U^i and U^j in the above variations, it would also be possible to conclude that

$$\langle M, X_0(t_1)^{-1}[B_j, B_i]X_0(t_1)\rangle \geq 0.$$

Hence, for all pairs i, j, this inequality sign may be replaced by an equality sign. But this fact together with (13) implies that M is orthogonal to all elements in $\mathfrak{g}$, and this is only possible if

$$M = 0.$$

By Theorem 3.1, it is not possible for both ψ_0 and M to be zero, and this contradiction proves the theorem.

We shall assume for the remainder of this paper that

$$\mathfrak{g} = \{A, B_1, \ldots, B_m\}_{LA}$$

is a subalgebra of

$$\{C \in \mathfrak{gl}(n, \mathbb{R}) : {}^tCR + RC = 0\},$$

where R is some nonsingular symmetric or skew-symmetric matrix such that

$$R^2 = \pm I.$$

Moreover, let us suppose that $\mathfrak{g}$ is closed under matrix transposition, i.e., $C \in \mathfrak{g}$ implies

$${}^tC \in \mathfrak{g}.$$

With

$$h_0 \subseteq \mathfrak{g}$$

denoting the linear span of $B_1, \ldots, B_m$, let

$$\tilde{Q} : h_0 \to h_0$$

be the linear operator such that

$$q_{ij} = \langle B_i, \tilde{Q}(B_j)\rangle.$$

Extend $B_1, \ldots, B_m$ to an orthonormal (with respect to $\langle \cdot, \cdot \rangle$) basis $B_1, \ldots B_m, B_{m+1}, \ldots, B_\nu$, for $\mathfrak{g}$. It is evident that ${}^tB_1, \ldots, {}^tB_\nu$ is also an orthonormal basis. The next theorem is a major extension of the results reported in Ref. 1 and Ref. 4.

Theorem 3.4. Together with the assumptions and notation set out in the preceding paragraph, make the additional assumption that the optimization problem of Section 1 is normal. Let

$$U^0(t) = \sum_{i=1}^{m} u_i^0(t) B_i$$

be an optimal control, and let $X_0(t)$ be the corresponding optimal trajectory. There exists a constant $M \in \mathfrak{g}$ (the negative transpose of the M in Theorem 3.1) and differentiable functions $v_{m+1}(\cdot), \ldots, v_\nu(\cdot)$ such that

$$X_0(t) M X_0(t)^{-1} = {}^t\left\{\tilde{Q}\left(\sum_{i=1}^{m} u_i^0(t) B_i\right)\right\} + \sum_{i=m+1}^{\nu} v_i(t) {}^tB_i; \tag{14}$$

moreover, the differential equation

$$\left(\frac{d}{dt}\right)\left\{\tilde{Q}\left(\sum_{i=1}^{m} u_i(t) B_i\right) + \sum_{i=m+1}^{\nu} u_i(t) B_i\right\}$$
$$= \left[\tilde{Q} \sum_{i=1}^{m} u_i(t) B_i + \sum_{i=m+1}^{\nu} u_i(t) B_i, {}^tA + \sum_{i=1}^{m} u_i(t) {}^tB_i\right] \tag{15}$$

is satisfied by

$$u_i(t) = u_i^0(t), \qquad i = 1, \ldots, m,$$

and

$$u_i(t) = v_i(t), \qquad i = m+1, \ldots, \nu.$$

Proof. It has been seen that, under the assumption of normality, Theorem 3.2 implies the existence of an $M_0 \in \mathfrak{g}$ such that

$$\langle M_0, X_0(t)^{-1} B_i X_0(t) \rangle = -\sum_{j=1}^{m} q_{ij} u_j^0(t).$$

In the light of the special structure that we have assumed for $\mathfrak{g}$, the left-hand side of this equation may be rewritten as

$$\langle X_0(t) {}^tM_0 X_0(t)^{-1}, {}^tB_i \rangle.$$

Hence,

$$\sum_{j=1}^{m} q_{ij}u_j^0(t) = \langle X_0(t)MX_0(t)^{-1}, {}^tB_i\rangle, \qquad i=1,\ldots,m,$$

where

$$M = -{}^tM_0.$$

Having chosen $B_{m+1},\ldots,B_\nu$ as above, define

$$v_i(t) = \langle X_0(t)MX_0(t)^{-1}, {}^tB_i\rangle, \qquad i=m+1,\ldots,\nu.$$

Then, comparing the inner products of the right-hand and left-hand sides of (14) with ${}^tB_1,\ldots,{}^tB_\nu$ successively, (14) follows.

Equation (15) is a straightforward calculation using (14).

In the case that the B_i span $\mathfrak{g}$, Eq. (15) reduces to (4). We shall also refer to (15) as the *Euler–Lagrange equation* for the optimal control.

4. Solutions to the Euler–Lagrange Equation

In this section, we shall consider systems of the form (1) which evolve on a matrix Lie group G with corresponding Lie algebra $\mathfrak{g}$ and for which

$$\mathfrak{g} \subseteq h_0 + [h_0, h_0],$$

where h_0 is the linear span of the B_i's. By Theorem 3.3, the optimization problem for such a system is normal. It is therefore possible to compute the optimal control by solving the differential equation (15). This unfortunately is not a trivial matter in general. Example 2.1 with

$$q_1 \neq q_2 \neq q_3 \neq q_1$$

presents a simple case in which it is known that elliptic functions must be introduced to solve the Euler–Lagrange equation. Nevertheless, there are a number of cases in which (15) may be solved easily in terms of elementary functions.

Let us assume that the collection of matrices $A, B_1,\ldots,B_m$ is self-contragredient, i.e.,

$$A = -{}^tA, \qquad B_i = -{}^tB_i.$$

As above, let $B_{m+1},\ldots,B_\nu$ be chosen so that the $B_1,\ldots,B_\nu$ form an orthonormal basis for $\mathfrak{g}$. Suppose that $\mathfrak{g}$ has an orthogonal direct sum decomposition

$$\mathfrak{g} = k_0 \oplus k_1 \oplus \cdots \oplus k_r \oplus k_{r+1} \oplus \cdots \oplus k_s,$$

such that, for $i=0, 1, \ldots, r$, k_i is a subspace and, for $i=r+1, \ldots, s$, k_i is a subalgebra. Moreover, suppose that

$$[k_i, k_j]=0 \qquad \text{for } i\neq j, \qquad r+1\leq i, j\leq s,$$

$$k_i\subseteq[k_r, k_r], \qquad i=r+1, \ldots, s,$$

$$k_i\subseteq[k_{i-1}, k_{i-1}], \qquad i=1, \ldots, r.$$

Finally, let us assume that $[k_i, k_j]\subseteq k_i$ for $0\leq i<j\leq r$. As usual $P_i: \mathfrak{g}\to k_i$ denotes the orthogonal projection.

Lemma 4.1. For $i=0, \ldots, r$ and $j=r+1, \ldots, s$,

$$[k_i, k_j]\subseteq k_i.$$

Proof. For any $X, Y, Z\in \mathfrak{g}$,

$$\langle[X, Y], Z\rangle=\langle X, [Y, Z]\rangle. \tag{16}$$

Suppose that

$$X\in k_i, \qquad Y\in k_j, \qquad Z\in k_k,$$

where

$$0\leq i\leq r \qquad \text{and} \qquad r+1\leq j, k\leq s.$$

If

$$k=j,$$

the right-hand member of (16) is zero, since k_i and k_j are assumed to be orthogonal. If

$$k\neq j,$$

the right-hand side of (16) is zero, since

$$[k_k, k_j]=0 \qquad \text{for } k\neq j.$$

Finally, if

$$0\leq k\leq r, \qquad k\neq i,$$

then

$$[Y, Z]\in k_k,$$

and again the right-hand side of (16) vanishes, since k_i and k_k are orthogonal. Thus,

$$\langle[X, Y], Z\rangle=0$$

if

$$Z \in k_k \qquad \text{with } k \neq i,$$

and this proves the lemma.

Lemma 4.2. Let

$$X, Y \in k_{r+1} \oplus \cdots \oplus k_s.$$

Then,

(i) $\quad [X, P_j(Y)] = [P_j(X), P_j(Y)], \qquad j = r+1, \ldots, s;$

(ii) $\exp(X) \cdot P_j(Y) \cdot \exp(-X) = \exp(P_j(X)) \cdot P_j(Y) \cdot \exp(-P_j(X)),$
$\qquad j = r+1, \ldots, s.$

Proof. (i) follows easily from Lemma 4.1. For the proof of (ii), note first that

$$\exp(X) = \exp\left(\sum_{i=r+1}^{s} P_i(X)\right) = \prod_{i=r+1}^{s} \exp(P_i(X)),$$

which follows since

$$[P_i(X), P_j(X)] = 0 \qquad \text{for } i \neq j.$$

Then, we find that

$$\exp(P_i(X)) \cdot P_j(Y) \cdot \exp(-P_i(X)) = \begin{cases} P_j(Y), & i \neq j, \\ \exp(P_j(X)) \cdot P_j(Y) \cdot \exp(-P_j(X)), & i = j, \end{cases}$$

and indeed

$$\prod_{i=r+1}^{s} \exp(P_i(X)) \cdot P_j(Y) \cdot \prod_{i=r+1}^{s} \exp(-P_i(X)) = \exp(P_j(X)) \cdot P_j(Y) \cdot \exp(-P_j(X)).$$

This proves Lemma 4.2.

Suppose that h_0 may be written with respect to our direct sum decomposition as

$$h_0 = k_0 \oplus \cdots \oplus k_{r-1} \oplus k_{r+1} \oplus \cdots \oplus k_{s-1}. \tag{17}$$

The terms k_r and k_s have been omitted; the sum

$$k_r \oplus k_s = h_0^{\perp}.$$

Assume that the linear mapping

$$\tilde{Q} : h_0 \to h_0$$

appearing in (15) is of the special form

$$\tilde{Q}(H) = \lambda_i H,$$

if $H \in k_i$; λ_i is a nonzero real number. Let

$$U_i(t) = P_i\left[\sum_{j=1}^{\nu} u_j(t)B_j\right], \qquad i = 0, 1, \ldots, s.$$

Then, taking

$$\lambda_r = \lambda_s = 1,$$

(15) may be rewritten as

$$\sum_{i=0}^{s} \lambda_i (d/dt)U_i(t) = \left[A + \sum_{i=0}^{r-1} U_i(t) + \sum_{i=r+1}^{s-1} U_i(t), \sum_{j=0}^{s} \lambda_j U_j(t)\right].$$

By assumptions placed on the k_i, this may be rewritten as

$$\begin{aligned}\sum_{i=0}^{s} \lambda_i (d/dt)U_i(t) = \sum_{i=0}^{s} \lambda_i [A, U_i(t)] &+ \sum_{0 \leq i < j \leq r-1} (\lambda_j - \lambda_i)[U_i(t), U_j(t)] \\ &+ \sum_{i=0}^{r-1} \sum_{j=r+1}^{s-1} (\lambda_j - \lambda_i)[U_i(t), U_j(t)] \\ &+ \sum_{i=0}^{r-1} [U_i(t), U_r(t) + U_s(t)] \\ &+ \sum_{i=r+1}^{s-1} [U_i(t), U_r(t)].\end{aligned}$$

If

$$P_0(A) = P_1(A) = \cdots = P_r(A) = 0,$$

this equation may be decoupled as follows:

$$(d/dt)U_i(t) = [A, U_i(t)], \qquad i = r+1, \ldots, s,$$

$$(d/dt)U_r(t) = [A, U_r(t)] + \left[\sum_{i=r+1}^{s-1} U_i(t), U_r(t)\right],$$

$$\begin{aligned}(d/dt)U_{r-1}(t) = [A, U_{r-1}(t)] &+ \sum_{j=r+1}^{s-1} (\lambda_{r-1} - \lambda_j)/\lambda_{r-1}[U_j(t), U_{r-1}(t)] \\ &- \lambda_{r-1}^{-1}[U_r(t) + U_s(t), U_{r-1}(t)],\end{aligned}$$

$$\begin{aligned}(d/dt)U_i(t) = [A, U_i(t)] &+ \sum_{j=i+1}^{r-1} (\lambda_i - \lambda_j)/\lambda_i[U_j(t), U_i(t)] \\ &+ \sum_{j=r+1}^{s-1} (\lambda_i - \lambda_j)/\lambda_i[U_j(t), U_i(t)] \\ &- \lambda_i^{-1}[U_r(t) + U_s(t), U_i(t)], \qquad i = 0, 1, \ldots, r-2.\end{aligned}$$

It is evident that certain of these equations may be explicitly solved. To begin, for $i = r+1, \ldots, s$, we have

$$U_i(t) = \exp(At) \cdot U_i(0) \cdot \exp(-At).$$

From this, the equation for $U_r(t)$ becomes

$$(d/dt)U_r(t) = \left[A + \exp(At) \cdot \left(\sum_{i=r+1}^{s-1} U_i(0)\right) \cdot \exp(-At), U_r(t)\right],$$

so that

$$U_r(t) = \exp(At) \cdot \exp(C_1 t) \cdot U_r(0) \cdot \exp(-C_1 t) \cdot \exp(-At),$$

where

$$C_1 = \sum_{i=r+1}^{s-1} U_i(0).$$

The equation for $U_{r-1}(t)$ may now be written as

$$\begin{aligned}(d/dt)&U_{r-1}(t)\\ &= [A + \exp(At) \cdot \exp(C_1 t) \cdot C_2 \cdot \exp(-C_1 t) \cdot \exp(-At), U_{r-1}(t)],\end{aligned}$$

where

$$C_2 = \sum_{j=r+1}^{s-1} (\lambda_{r-1} - \lambda_j)/\lambda_{r-1} U_j(0) - \lambda_{r-1}^{-1}(U_r(0) + U_s(0)).$$

It is easy to see that

$$\begin{aligned}U_{r-1}(t) = \exp(At) \cdot \exp(C_1 t) \cdot \exp(C_2 - C_1)t \cdot U_{r-1}(0)&\\ \cdot \exp(C_1 - C_2)t \cdot \exp(-C_1 t) \cdot \exp(-At).&\end{aligned}$$

For $i = 0, 1, \ldots, r-2$, the differential equations for $U_i(t)$ do not admit solutions of this simple form. We collect these results in the following theorem.

Theorem 4.1. Suppose that h_0 and $h_0^\perp$ have the above orthogonal direct sum decompositions, and let

$$r = 1$$

in (17). Suppose, moreover, that

$$\tilde{Q} : h_0 \to h_0$$

has the form assumed above. Then, the solution to (15) may be written as

$$\begin{aligned}\sum_{i=1}^{m} u_i^0(t)B_i &= U_0(t)+\sum_{j=2}^{s-1} U_j(t)\\ &= \exp(At)\cdot C_1\cdot \exp(-At)\\ &\quad +\exp(At)\cdot \exp(C_1t)\cdot \exp(C_2-C_1)t\cdot U_0(0)\exp(-C_1t)\cdot \exp(-At),\\ \sum_{i=m+1}^{\nu} v_i(t)B_i &= U_r(t)+U_s(t)\\ &= \exp(At)\cdot \exp(C_1t)\cdot (U_r(0)+U_s(0)\cdot \exp(-C_1t)\cdot \exp(-At),\end{aligned}$$

where C_1 and C_2 are given above.

Corollary 4.1. Suppose that the assumptions of the preceding theorem are operative, and suppose also that

$$k_1 \equiv \{0\}.$$

Then, the solution to (15) may be written as

$$\begin{aligned}\sum_{i=1}^{m} u_i^0(t)B_i &= \sum_{j=0}^{s-1} U_j(t)\\ &= \exp(At)\cdot \exp(C_2t)\cdot \sum_{j=0}^{s-1} U_j(0)\cdot \exp(-C_2t)\cdot \exp(-At), \qquad (18)\end{aligned}$$

while

$$\sum_{i=m+1}^{\nu} v_i(t)B_i = U_s(t) = \exp(At)\cdot U_s(0)\cdot \exp(-At).$$

The trajectory $X_0(t)$ corresponding to this via (1) is given by

$$X_0(t) = \exp(At)\cdot \exp(C_2t)\cdot \exp(U(0)-C_2)t,$$

where

$$U(0) = \sum_{j=0}^{s-1} U_j(0).$$

Proof. If

$$k_1 = \{0\},$$

we have

$$U_1(0) = 0,$$

and hence

$$C_2 = \sum_{j=2}^{s-1} (\lambda_0 - \lambda_j)/\lambda_0 U_j(0) - \lambda_0^{-1} U_s(0).$$

From this, we see that

$$[C_1, C_2] = 0,$$

and we compute $\sum_{i=1}^{m} u_i^0(t)B_i$ and $\sum_{i=m+1}^{\nu} v_i(t)B_i$ from Theorem 4.1. The expression for $X_0(t)$ is easily seen to be correct by differentiating.

Example 4.1. Consider the following evolution equation and performance measure for a system evolving on $SU(3)$:

$$(d/dt)X(t) = \begin{bmatrix} 0 & a_1 + ia_2 & -u_2(t) + iu_4(t) \\ -a_1 + ia_2 & 0 & u_1(t) + iu_3(t) \\ u_2(t) + iu_4(t) & -u_1(t) + iu_3(t) & 0 \end{bmatrix} X(t),$$

$$\eta = \int_0^T \sum_{i=1}^{4} u_i(t)^2 \, dt.$$

The Lie algebra $su(3)$ may be embedded in the Lie algebra $so(6)$ via the realification homomorphism

$$C + iD \overset{\rho}{\rightarrow} \begin{bmatrix} C & D \\ -D & C \end{bmatrix}.$$

Hence, we may consider the corresponding equation

$$(d/dt)Y(t) = \left(\rho(A) + \sum_{i=1}^{4} u_i(t)\rho(B_i)\right) Y(t)$$

evolving on the image group

$$\{\exp\{\rho(A), \rho(B_1), \rho(B_2), \rho(B_3), \rho(B_4)\}_{LA}\}_G.$$

Here,

$$A = \begin{bmatrix} 0 & a_1 + ia_2 & 0 \\ -a_1 + ia_2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \qquad B_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{bmatrix}, \qquad B_4 = \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix}.$$

The mapping

$$\tilde{Q}: h_0 \to h_0,$$

where h_0 = linear span of $\rho(B_1), \rho(B_2), \rho(B_3), \rho(B_4)$, is $\frac{1}{4}I$, where I is the identity mapping on the vector space h_0. The images under ρ of

$$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} i & 0 & 0 \\ 0 & -\frac{1}{2}i & 0 \\ 0 & 0 & -\frac{1}{2}i \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{bmatrix},$$

form a basis in $\rho(su(3))$ for $h_0^{\perp}$. It is easy to verify that $h_0^{\perp}$ is a subalgebra and

$$h_0^{\perp} \subseteq [h_0, h_0],$$

so that, in terms of the decomposition (17), we have

$$r = 1, \qquad s = 2, \qquad k_0 = h_0, \qquad k_1 = \{0\}, \qquad k_2 = h_0^{\perp}.$$

From Theorem 4.1, we find that there exist functions $v_1(t), v_2(t), v_3(t), v_4(t)$ which, together with the optimal controls, satisfy the equation

$$(d/dt)\begin{bmatrix} iv_3(t) & v_1(t)+iv_2(t) & -u_2(t)+iu_4(t) \\ -v_1(t)+iv_2(t) & iv_4(t)-\frac{1}{2}iv_3(t) & u_1(t)+iu_3(t) \\ u_2(t)+iu_4(t) & -u_1(t)+iu_3(t) & -iv_4(t)-\frac{1}{2}iv_3(t) \end{bmatrix}$$

$$= \left[\begin{bmatrix} 0 & a_1+ia_2 & -u_2(t)+iu_4(t) \\ -a_1+ia_2 & 0 & u_1(t)+iu_3(t) \\ u_2(t)+iu_4(t) & -u_1(t)+iu_3(t) & 0 \end{bmatrix},\right.$$

$$\left.\begin{bmatrix} iv_3(t) & v_1(t)+iv_2(t) & -u_2(t)+iu_4(t) \\ -v_1(t)+iv_2(t) & iv_4(t)-\frac{1}{2}iv_3(t) & u_1(t)+iu_3(t) \\ u_2(t)+iu_4(t) & -u_1(t)+iu_3(t) & -iv_4(t)\frac{1}{2}iv_3(t) \end{bmatrix}\right]$$

This equation is obtained by pulling (15) back from $\rho(su(3))$ to $su(3)$. From Corollary 4.1, we find that the solution to this equation is of the form

$$\begin{bmatrix} 0 & 0 & -u_2^0(t)+u_4^0(t) \\ 0 & 0 & u_1^0(t)+u_3^0(t) \\ u_2^0(t)+u_4^0(t) & -u_1^0(t)+u_3^0(t) & 0 \end{bmatrix}$$

$$= \exp(At)\cdot\exp(Ct)\cdot\begin{bmatrix} 0 & 0 & -u_2^0(0)+u_4^0(0) \\ 0 & 0 & u_1^0(0)+u_3^0(0) \\ u_2^0(0)+u_4^0(0) & -u_1^0(0)+u_3^0(0) & 0 \end{bmatrix}$$

$$\cdot\exp(-Ct)\cdot\exp(-At),$$

where

$$C = \begin{bmatrix} iv_3(0) & v_1(0)+iv_2(0) & 0 \\ -v_1(0)+iv_2(0) & iv_4(0)-\frac{1}{2}iv_3(0) & 0 \\ 0 & 0 & -iv_4(0)-\frac{1}{2}iv_3(0) \end{bmatrix}.$$

Example 4.2. Consider the system on $S1(2, \mathbb{R})$

$$(d/dt)\begin{bmatrix} x_1(t) & x_2(t) \\ x_3(t) & x_4(t) \end{bmatrix} = \begin{bmatrix} 0 & u_1(t) \\ u_2(t) & 0 \end{bmatrix}\begin{bmatrix} x_1(t) & x_2(t) \\ x_3(t) & x_4(t) \end{bmatrix},$$

for which we have defined the performance measure

$$\eta = \int_0^T [u_1(t)^2 + u_2(t)^2]\, dt.$$

By Theorem 3.4, there exists a function $v(t)$ which, together with the optimal controls $u_1^0(t)$ and $u_2^0(t)$, satisfies the Euler–Lagrange equation

$$(d/dt)\begin{bmatrix} v(t) & u_1^0(t) \\ u_2^0(t) & -v(t) \end{bmatrix} = \left[\begin{bmatrix} v(t) & u_1^0(t) \\ u_2^0(t) & -v(t) \end{bmatrix}, \begin{bmatrix} 0 & u_2^0(t) \\ u_1^0(t) & 0 \end{bmatrix}\right].$$

Writing this out componentwise, we obtain

$$dv/dt = u_1^0(t)^2 - u_2^0(t)^2,$$

$$du_1/dt = 2v(t)u_2^0(t),$$

$$du_2/dt = -2v(t)u_1^0(t).$$

Making the change of variables

$$y_1(t) = -2v(t), \qquad y_2(t) = 2(u_1^0(t)+u_2^0(t)), \qquad y_3(t) = 2(u_1^0(t)-u_2^0(t)),$$

we obtain the equivalent system of equations

$$dy_1/dt = -\tfrac{1}{2}y_2(t)y_3(t),$$

$$dy_2/dt = y_1(t)y_3(t),$$

$$dy_3/dt = -y_1(t)y_2(t),$$

and these are easily solved in terms of the elliptic functions of Jacobi (Ref. 14, Chapter 22).

5. Conclusions

In this paper, we have begun an investigation into methods of finding explicit solutions to nonlinear optimal control problems. Our central result is that, for quadratic performance measures and systems (1) evolving on certain matrix Lie groups, optimal controls are given as solutions to a certain quadratic matrix differential equation; in a number of cases, the form of the optimal controls may be determined explicitly in closed form from this differential equation. In a future paper, we shall show how the boundary conditions of the optimization problem may be incorporated to specify completely the optimal controls.

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