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Nonlinear Systems and Differential Geometry

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Abstract—Nonlinear systems which are governed by a finite number of ordinary differential equations with controls present constitute a large and important class of models for practical purposes. In the last few years, there has been considerable progress in our understanding of this class of models. This is an expository paper devoted to surveying and explaining some of the main results currently available. Background material on manifold theory is included in order to make the paper more nearly self-contained.

I. Introduction

THE PURPOSE of this paper is to describe some of the main theoretical results on the class of input/output (I/O) models which take the form

$$\dot{x}(t) = f[x(t)] + u(t)g[x(t)]; \quad y(t) = h[x(t)]; \quad x(0) = x_0. \tag{1}$$

The most significant results relate to a) controllability theory, b) I/O theory based on Volterra series, c) isomorphism theorem and bilinearization, and d) stochastic theory.

Our intention is to give an introduction to this area. We will give a number of the most important results but we do not give proofs unless there are special circumstances which make it desirable. The material to be discussed has a great deal of intuitive content but it also requires some technical developments. In fact, the solution of some of the central problems required the development of some new pure mathematics. In an attempt to present a blend of intuition and solid theory, we have included in an appendix the definition of many technical terms and also precise statements of a few background theorems.

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In spite of the rather general form of nonlinear systems we work with here, most of the results given here could be stated in even greater generality. For example, the restriction to one input is completely unnecessary; many results apply to $\dot{x} = f(x, u)$, etc. It seemed counterproductive to move in this direction, however, in a paper intended as an expository introduction.

We introduce a few standard notations. R^n stands for n-dimensional Cartesian space with its ordinary topology, if $x \in R^n$ then $\|x\| = (x_1^2 + x_2^2 + \cdots x_n^2)^{1/2}$. We use letters such as M and N for differentiable manifolds (see appendix for definition). We always assume that $f(\cdot)$, $g(\cdot)$, and $h(\cdot)$ in (1) are infinitely differentiable and often we assume even more, that they depend analytically on their arguments. We call systems of this latter type *linear-analytic* systems.

The special case of bilinear systems

$$\dot{x}(t) = Ax(t) + u(t)Bx(t); \quad y(t) = \langle c, x(t) \rangle : x(0) = x_0$$

has received a great deal of attention in the literature in the past few years. Although this theory is, itself, quite a successful generalization of the linear theory, it now appears that it should be viewed in the context of the more general class of systems described here. The reasons for this are that many of the principle results on bilinear systems are true for this wider class of systems and the class of bilinear systems has the unfortunate property of not being closed under composition and feedback. This is not to say that bilinear systems are of little interest. On the contrary, they are the most interesting specialization of this theory, and, in fact, they appear from the point of view of the theory of linear-analytic systems to be even more general, and hence of greater interest, than was first expected.

With regard to the literature, there is a recent conference proceedings [1] which contains papers on most of the topics to be discussed here. In particular, the paper by Lobry is relevant for Section II on controllability. The paper by Sussmann is appropriate for Section IV. The papers by Clark and Elliott will provide the details behind the material discussed in Section IV. Specific citations will be given in the appropriate places.

II. Controllability

We begin with a discussion of a result of Frobenius. This theorem is fundamental to the geometric theory of control. In fact, one could say that it is to control theory what the Cauchy-Lipschitz existence theorem for ordinary differential equations is to the theory of autonomous models found in analytical mechanics, electrical circuit theory, etc.

Suppose we have a single differential equation in R^n

$$\dot{x}(t) = f[x(t)]; \quad x(0) = x_0 \in R^n \tag{2}$$

with f smooth enough to define a unique solution. In the theory of differential equations, one ordinarily indicates the value of the solution at time t, by $\phi(t, x_0)$ unless f is linear in which case the exponential function is displayed. It is more common in differential geometry to write $(\exp tf)x_0$ instead of $\phi(t, x_0)$ to indicate the solution of (2) regardless of whether f is linear or not. We will see that the disadvantage of the slight ambiguity of this notation is offset by a savings in the number of parentheses one must use. One way to describe the usual existence theorem of Cauchy-Lipschitz is to say that there exists a one-dimensional subset of R^n, $M = \{x : x = (\exp tf)x_0;\ |t| < \epsilon\}$ such that $f[x]$ is tangent to M at each point. (A glance at the appendix at this point is recommended for those who want a definition of a manifold, vector fields on manifolds, etc.) Stated in this way, the Cauchy-Lipschitz theorem answers a special case of the following more general question. Consider

$$\dot{x}(t) = \sum_{i=1}^{m} u_i(t) f_i[x(t)], \qquad x(0) = x_0 \in R^n. \tag{3}$$

Under what circumstances can we find a smooth p-dimensional subset M of R^n such that the $f_i(x)$ span the tangent space of M at each point? Why is this question of any interest? Well if the u_i's are controls which can be turned on and off, reversed, etc., at will, and if such an M exists, then apparently x will be able to move anywhere in M but not out of M.

At the first level of complexity, the existence of M hinges on the following observation. If we follow the integral curve of $\dot{x} = f_1(x)$ for t units of time, then $\dot{x} = f_2(x)$ for t units, then $\dot{x} = -f_1(x)$ for t units and for t units of time and $\dot{x} = -f_2[x]$ for t units of time then the resulting point is $(\exp -tf_2)(\exp -tf_1)(\exp tf_2)(\exp tf_1)x_0$ (see Fig. 1). Working out the value of this product based on repeated use of the second-order expansion

$$x(t) = x_0 + tf_i(x_0) + \left(\frac{t^2}{2}\right)\frac{\partial f_i}{\partial x} f_i(x_0) + 0(t^3)$$

yields

$$\exp(-tf_2)(\exp -tf_1)(\exp tf_2)(\exp tf_1)x_0 = x_0 + (t^2/2)[f_2, f_1]x_0 + 0(t^3)$$

where

$$[f_2, f_1] = \frac{\partial f_2}{\partial x} f_1 - \frac{\partial f_1}{\partial x} f_2$$

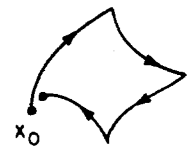

Fig. 1. Motivating the definition of ubiquitous bracket of Lie.

is the so-called *Lie bracket* (also sometimes called the Jacobi bracket, also sometimes defined with the opposite sign) of the vector fields f_1 and f_2. This calculation, which "everyone should do once in his life" is most significant. Everything else depends on it. If $[f_1, f_2]$ is not a linear combination of $f_1, f_2, \cdots, f_m$ then $[f_1, f_2]$ represents a "new" direction in which the solution can move and the original problem of finding a manifold such that $f_1, f_2, \cdots, f_m$ span the tangen space will not be solvable.

We introduce a definition which is useful when we want to rule out this behavior. One calls a set of vector fields $\{f_1, f_2, \cdots, f_m\}$ *involuative* if there exists γ_{ijk} such that

$$[f_i, f_j](x) = \sum_{k=1}^{m} \gamma_{ijk}(x) f_k(x).$$

Some thought about the above construction should persuade the reader that the property of being involuative is necessary in order to be able to "integrate" the set of vector fields $(f_1, f_2, \cdots, f_m)$ to get a "solution" manifold. More than 60 years ago, Frobenius established his theorem on "complete integrability" which showed that, under some mild regularity assumptions, this condition is sufficient as well. Frobenius [2] stated his result as a local result (and actually a "dual" to the form we discuss here) but subsequently it has beep established in a global form which asserts the existence of maximal (with respect to theoretic inclusion) solutions.

Just as there are many versions of the existence theorem for ordinary differential equations, there are many versions of the Frobenius theorem. We give two examples which play a role in the control literature.

Theorem 1: (Versions of the Frobenius theorem.) Let $\{f(x_1), f(x_2), \cdots, f(x_n)\}$ be an involuative collection of vector fields which are

a) analytic on an analytic manifold M. Then given any point $x_0 \in M$ there exists a maximal submanifold N containing x_0 such that $\{f_i(x)\}$ spans the tangent space of N at each point of N.
b) C^∞ on a C^∞ manifold M with the dimension of the spa of $\{f_i(x)\}$ being constant on M. Then given any point $x_0 \in M$ there exists a maximal submanifold N containing x_0 such that $\{f_i(x)\}$ spans the tangent space of N at each point of N.

Example 1: Consider the vector field on R defined by the function x. R is an analytic manifold and this vector field is analytic. According to a) for each point in R, there is a maximal submanifold of R such that x spans the tangent space of this submanifold at each point. In this case, we see that there are three distinct submanifolds of R which arise in this way. If x_0 is positive, then N is the positive half-line $(0, \infty)$; if x_0 is negative, N is the negative half-line $(-\infty, 0)$; and if $x_0 = 0$, it is the zero dimensional manifold consisting of 0 alone.

Example 2: We give a second example with a little more geometric content. Consider three analytic vector fields in

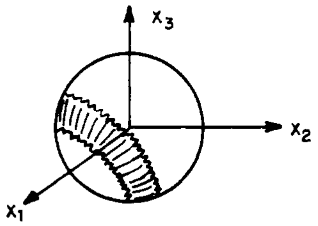

Fig. 2. Maximal integral manifold for the example.

R^3 represented by

$$f_1(x) = \begin{bmatrix} 0 \\ x_3 \\ -x_2 \end{bmatrix} \qquad f_2(x) = \begin{bmatrix} -x_3 \\ 0 \\ x_1 \end{bmatrix} \qquad f_3(x) = \begin{bmatrix} x_2 \\ -x_1 \\ 0 \end{bmatrix}.$$

This collection is involutive since $[f_1, f_2] = f_3$, $[f_2, f_3] = f_1$ and $[f_3, f_1] = f_2$, and at each point in $R^3 - \{0\}$ it contains exactly two linearly independent vectors ($x_1 f_1 + x_2 f_2 + x_3 f_3 = 0$). If we look at any nonzero point in R^3, say $x = \frac{1}{3}(\sqrt{3}, \sqrt{3}, \sqrt{3})$, then we can integrate this distribution through that point. For the given point, we get the set

$$N = \{x : \|x\| = 1\}$$

as the solution of this integral manifold problem. Why this set? If we integrate along one of the vector fields, say f_1, the integral curve is a circle; x_1 is constant, $x_2^2 + x_3^2$ is constant (see Fig. 2). Now given any point on the circle, we can integrate along $\dot{x} = \pm f_3$ to get a band around this circle, etc. In fact, this is how one sometimes proves the Frobenius theorem: riding successive vector fields to fill out a neighborhood of a given point. Notice that, in this case, the vectors f_1, f_2, $_3$, are, in fact, tangent to the spherical shell N at each point as required.

The relevance of the Lie bracket and Frobenius' theorem for controllability studies comes in via a theorem of Chow [3] and its refinement by others [4]-[8]. Given a control system in R^n

$$\dot{x}(t) = \sum_{i=1}^{m} u_i(t) f_i[x(t)], \qquad x(0) = x_0$$

we can certainly reach any point of the form

$$x = (\exp t_1 f_{\alpha_1})(\exp t_2 f_{\alpha_2})(\exp t_3 f_{\alpha_3}) \cdots (\exp t_m f_{\alpha_m}) x_0$$

simply by letting all but one of the inputs be zero with the remaining one being one. In particular, given two vector fields f_1 and f_2 we can reach

$$x = (\exp -tf_1)(\exp -tf_2)(\exp tf_1)(\exp tf_2)x_0$$
$$= x_0 + \tfrac{1}{2} t^2 [f_1, f_2] + 0(t^3).$$

This suggests that we can move in the direction $[f_1, f_2]$ even though this direction may not be in the linear span of the $\{f_i\}$. To push this idea to its logical conclusion, we need the following definition.

If $\{f_i\}$ is a collection of C^∞ or analytic vector fields on a manifold M, then we say that $\{f_i\}$ is a *Lie algebra of vector fields* provided that

i) $\{f_i\}$ is a real vector space with respect to ordinary vector addition and scalar multiplication,
ii) if f_j and f_k belong to $\{f_i\}$, then the Lie bracket $[f_i, f_j]$ belongs to $\{f_i\}$.

We say that the Lie algebra is finite dimensional if the real vector space $\{f_i\}$ is finite dimensional. For example, the collection of all polynomials form an infinite-dimensional Lie algebra of vector fields on R^1 since $[x^p, x^q] = (p - q)x^{p+q-1}$. On R^n, the set of all vector fields of the form Ax with A a constant matrix forms an n^2-dimensional Lie algebra since $[Ax, Bx] = (AB - BA)x$—thus giving closure under Lie bracket—and since the space of n by n matrices is an n^2-dimension vector space.

One can see the reason for introducing the Lie algebra in controllability problems. The previous calculation strongly suggests that not only is $(\exp tf_1)x_0$ and $(\exp tf_2)x_0$ in the reachable set from x_0 but also $(\exp t[f_1, f_2])x_0$. In fact, iterating on this theme suggests that $\{\exp tf\}x_0$ is in the reachable set if f can be expressed as a bracketed combination of the given f_i. Let $\{f_i\}_{LA}$ denote the Lie algebra of vector fields generated by $\{f_i\}$—that is, take all linear combinations of elements of $\{f_i\}$, take Lie brackets, take all linear combinations, take Lie brackets, etc. to arrive at the smallest Lie algebra of vector fields which contains $\{f_i\}$ and call this $\{f_i\}_{LA}$.

Suppose that the vector fields in $\{f_i\}_{LA}$ are *complete*, i.e., that $(\exp tf)x$ is defined for all $-\infty < t < \infty$. (If the manifold is R^n, this means no finite escape times in forwards or backwards time.) In this case, there is a group of mappings of M into itself which is closely connected with $\{f_i\}_{LA}$ and which is obtained by "exponentiating" $\{f_i\}_{LA}$. We explain this as follows. The set of all C^∞ one to one and onto mappings of a C^∞-manifold onto itself, having the property that the inverse mappings are also C^∞, is called the group of diffeomorphisms of M and is written diff (M). The set of such maps is clearly closed under inversion and composition, justifying the label group. Given f, for each t $(\exp tf)$ defines a map of M into itself which is just the mapping produced by the flow on M defined by the differential equation $\dot{x} = f(x)$. We denote the smallest subgroup of diff (M) which contains $\exp tf$ for all f in $\{f_i\}$ by $\{\exp \{f_i\}\}_G$. It is clear that any point x in M of the form $x = \{\exp \{f_i\}\}_G x_0$ can be reached from x_0 along solution curves of (3) because in this case

$$x = (\exp t_1 f_{\alpha_1})(\exp t_2 f_{\alpha_2}) \cdots (\exp t_m f_{\alpha_m}) x_0$$

and piecewise constant controls suffice to carry out the transfer.

A question of major concern can now be asked. What is the difference between $\{\exp \{f_i\}\}_G x_0$ and $\{\exp \{f_i\}_{LA}\}_G x_0$? The former is reachable and the latter has the appearance of something larger. The theorem of Chow says that these sets are, in fact, equal under rather weak assumptions.

Theorem 2 (Versions of Chow's Theorem): Let $\{f_1(x), f_2(x), \cdots, f_m(x)\}$ be a collection of vector fields such that the collection $\{f_1(x), f_2(x), \cdots, f_m(x)\}_{LA}$ is

a) analytic on an analytic manifold M. Then given any point $x_0 \in M$, there exists a maximal submanifold $N \subset M$ containing x_0 such that $\{\exp \{x_i\}\}_G x_0 = \{\exp \{x_i\}_{LA}\}_G x_0 = N$.
b) C^∞ on a C^∞ manifold M with dim (span $\{f_i(x)\}_{LA}$) constant on M. Then given any point $x_0 \in M$, there exists a maximal submanifold $N \subset M$ containing x_0 such that $\{\exp \{x_i\}\}_G x_0 = \{\exp \{x_i\}_{LA}\}_G x_0 = N$.

Clearly this result goes a very long way toward answering the controllability question for the system in (3). Most often in applications, however, we are interested in systems which have a drift term, i.e., systems of the form

$$\dot{x}(t) = f[x(t)] + u(t)g[x(t)], \qquad x(0) = x_0. \qquad (4)$$

As a result about controllability of this class of systems, the main limitation on Chow's theorem is that it does not distinguish between positive and negative time. That is, the submanifold whose existence is guaranteed by Theorem 2 may include points which can *only* be reached by passing backwards along the vector field $f(x)$. This means that while the reachable set from x_0 will always be contained in the manifold defined by Chow's theorem, in general, it will be a proper subset of this manifold. The most affirmative statement is the following result proved at various levels of generality by Krener [7], Lobry [6], and Sussmann and Jurdjevic [8].

Theorem 3: Suppose f and g are vector fields on a manifold M and suppose that $\{f, g\}$ meets either of the conditions for Chow's theorem. Then the reachable set for (4) contains an open subset of the manifold $N = \{\exp \{f, g\}_{LA}\}_G x_0$.

A slick proof of this theorem is given by Krener (see also Lobry's survey in [1]). This theorem underscores the relatively meager state of our knowledge regarding the reachable set of points for systems with a drift term.

Example: Consider the problem in R^2:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = v(t) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + u(t) \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

if $v(t)$ and $u(t)$ are arbitrary, then an application of Chow's theorem shows that any point in $R^2 - \{0\}$ can be steered to any point in $R^2 - \{0\}$. On the other hand, if $v(t)$ is constrained to be one, giving us a drift term, then the reachable set is difficult to describe precisely. We can say that if $x_1(0) > 0$ and $x_2(0) > 0$, then $x_1(t)$, $x_2(t)$ will lie in the positive quadrant for all $t > 0$ so that the reachable set is not all of $R^2 - \{0\}$.

In order to be able to present some interesting examples, we state a rather special circumstance under which the drift term causes no difficulty.

Theorem 4: Suppose that f and g are vector fields on a manifold M. Suppose that $\{f, g\}$ meet either of the conditions of Chow's theorem and suppose that for each initial condition x_0 the solution of

$$\dot{x}(t) = f[x(t)]$$

is periodic with a least period $T(x_0) < M$. Then the reachable set from x_0 for (4) is $\{\exp \{f, g\}_{LA}\}_G x_0$.

Proof (Sketch): The idea is that if we are at x_0 and we want to pass backwards along the vector field $f(x)$, we simply let u be zero and allow the free periodic motion to bring x_0 nearly back to x_0 along the integral curve of $\dot{x} = f(x)$. If the least period of the periodic solution through x_0 is T, then, by following $\dot{x} = f(x)$ for $T - \epsilon$ units of time, we have the same effect as following $\dot{x} = -f(x)$ for ϵ units of time. Thus we can, given enough time, reach any point which is reachable for

$$x(t) = v(t) f[x(t)] + u(t)g[x(t)].$$

We now consider some applications to illustrate the ideas and to contrast the theory with the well-known linear controllability theory. Our first example can be viewed as a Lie theoretic explanation of parametric amplification and parametric instability.

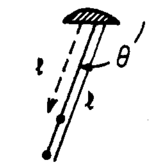

Fig. 3. Controllability approach to parametric amplification.

We consider a particular model for a child pumping a swing. The question being that of determining if controllability theory can predict the existence of pumping modes which will maintain or increase a given amplitude of oscillation (see Fig. 3). The model consists of a mass which moves up and down along a weightless rod (the child changes its center of mass). We ignore, to cut down the length of the formulae, the mass of swing itself. The state space fo the problem is M $\{(\theta, \dot{\theta}, l) : 0 \leq \theta \leq 2\pi,\ \dot{\theta} \in R,\ 0 < l < l_0\}$. Setting the time rate of change of the angular momentum about the support equal to the torque due to gravity gives

$$\frac{d}{dt} [\dot{\theta}(t) l^2(t) m] + l(t) m \sin\theta(t) = 0.$$

Differentiating gives the system

$$\ddot{\theta}(t) + [2\dot{l}(t)/l(t)]\,\dot{\theta}(t) + [1/l(t)] \sin\theta(t) = 0.$$

Now letting $\dot{\theta} = \phi$ and regarding $\dot{l} = u$ as the control, we get a set of three equations

$$\begin{aligned} \dot{\theta} &= \phi \\ \dot{\phi} &= u2lm + [1/l] \sin\theta = - \begin{bmatrix} \phi \\ (1/l)\sin\theta \\ 0 \end{bmatrix} + u - \begin{bmatrix} 0 \\ 2(\dot{l}/l)\sin\theta \\ 1 \end{bmatrix} \\ \dot{l} &= u. \end{aligned}$$

Notice that with $u \equiv 0$, the solution is periodic so we may apply Theorem 3 in computing the reachable set (provided we stay away from a neighborhood of $\theta = \pi$ where the period of the free motion is not bounded). Let the right side be $f + ug$. Then the Lie brackets include

$$[f, g] = \begin{bmatrix} 2\phi/l \\ (1/l^2)\sin\theta \\ 0 \end{bmatrix} \text{ and } [f, [f, g]] = \begin{bmatrix} (1/l^2)\sin\theta \\ (\phi/l^2)\cos\theta \\ 0 \end{bmatrix}$$

and thus $f, g, [f, g]$, and $[f, [f, g]]$ span except at $\phi = 0$, $\theta = 0$, or π. That is, the system is controllable if we have a "start".

As a second example, consider the Euler equations for a rigid body

$$\begin{bmatrix} \dot{\omega}_1(t) \\ \dot{\omega}_2(t) \\ \dot{\omega}_3(t) \end{bmatrix} = \begin{bmatrix} \alpha\omega_2(t)\omega_3(t) \\ \beta\omega_1(t)\omega_3(t) \\ \gamma\omega_1(t)\omega_2(t) \end{bmatrix} + \begin{bmatrix} n_1(t) \\ n_2(t) \\ n_3(t) \end{bmatrix}, \qquad \alpha + \beta + \gamma = 0$$

where $\alpha = (I_2 - I_3)/I_1$, $\beta = (I_3 - I_1)/I_2$, and $\gamma = (I_1 - I_2)/I_3$ with I_i being the moment of inertia about the ith principal axis. Let these equations of motion be denoted by

$$\dot{\omega}(t) = f[\omega(t)] + n(t)$$

where the ω_i are the angular velocities and $I_i n_i$ are the applied torques—both referred to principal axes. Now suppose there is only one torque available, say, from a pair of gas jets, but that the torque vector is not necessarily aligned with a principal axis. That is, we have

$$n = u(t)g$$

for some constant vector g. Again, the motion with $u = 0$ is periodic (elliptic function theory!) thus mitigating the effect of the drift term.

If we linearize this system about $\omega_1 = \omega_2 = \omega_3 = 0$, then linear theory will predict a one dimensional reachable set, provided g is nonzero. We will see that the Lie theory, on the other hand, answers the problem precisely. We begin by computing some of the vector fields belonging to the Lie algebra of vector fields generated by f and g. Let a, b, and c denote the components of g. We have, for example,

$$[f,g] = \begin{bmatrix} \alpha(b\omega_3 + c\omega_2) \\ \beta(a\omega_3 + c\omega_1) \\ \gamma(a\omega_2 + b\omega_1) \end{bmatrix} \qquad [[f,g],g] = 2\begin{bmatrix} \alpha bc \\ \beta ac \\ \gamma ab \end{bmatrix}$$

and, selecting a third constant vector field from $\{f,g\}_{LA}$,

$$[[f,g],[[f,g],g]] = 2\begin{bmatrix} \alpha a(\gamma b^2 + \beta c^2) \\ \beta b(\gamma a^2 + \alpha c^2) \\ \gamma c(\beta a^2 + \alpha b^2) \end{bmatrix}.$$

To check controllability, we want to determine if we have, as yet, vector fields which span R^3. In particular, we ask if g, $[f,g]$ and $[[f,g],g]$ span. A calculation shows that

$$\det\left(g, \tfrac{1}{2}[f,g], \tfrac{1}{2}[[f,g],g]\right) = \det\begin{bmatrix} a & \alpha bc & \alpha a(\gamma b^2 + \beta c^2) \\ b & \beta ac & \beta b(\gamma a^2 + \alpha c^2) \\ c & \gamma ab & \gamma c(\beta a^2 + \alpha b^2) \end{bmatrix}$$
$$= \beta\gamma a^4(\beta c^2 - \gamma b^2) + \alpha\gamma b^4(\gamma a^2 - \alpha c^2) + \alpha\beta c^4(\alpha b^2 - \beta a^2).$$

We now analyze the case of the symmetric rigid body. In this case, we can assume $\gamma = 0$ and since physics demands that $\alpha + \beta + \gamma = 0$, we see that the determinant is simply $\pm\alpha^3 c^4 \cdot (b^2 + a^2)$. If this vanishes, then one of three things happens—either α is zero, c is zero, or a and b are both zero. The first possibility implies that β is zero, thus $f \equiv 0$ and the reachable set is one dimensional. The second implies, since $\gamma = 0$, that ω_3 is a constant regardless of u and the reachable set is not three dimensional. The third implies that $\omega_1^2 + \omega_2^2$ is a constant regardless of u, thus the reachable set is not three dimensional. All cases are incompatible with controllability. If none of these occur, then the system is controllable and thus this is a complete analysis of the symmetric case.

If γ is nonzero, then the analysis is apparently more tedious, and it is not clear that the particular Lie brackets displayed here tell the whole story. Of course, if they do not, then we must look at additional elements of $\{f,g\}_{LA}$.

We conclude this section with a discussion of one further type of result on controllability which has been investigated by Hirschorn [9]. The idea is, that under some circumstances, it should be possible to "factor out" the effect of the drift term for

$$\dot{x}(t) = f[x(t)] + u(t)g[x(t)], \qquad x(t) = x_0 \tag{5}$$

so as to be able to express the reachable set at time t as

$$R(t) = \{\exp L\}_G (\exp tf)x_0$$

where L is some Lie algebra of vector fields constituted from f and g but not necessarily $\{f,g\}_{LA}$. A particularly simple special case of this kind of result is expressed by the following.

Theorem 5: Let $L = \{f,g\}_{LA}$ and let L_0 be the smallest subalgebra of L which contains g and is closed under Lie bracketing with f. Suppose that for all h in L_0 we have $[h,g] = \alpha_h g$ for some constant α. Then

$$R(t) = \{\exp L_0\}_G \exp tfx_0.$$

Proof: See [9, p. 945, Corollary 1] and make the necessary modifications to cover the present setup.

Example [10, p. 277]: Consider the linear time invariant system

$$\dot{x}(t) = Ax(t) + bu(t); \qquad x(0) \in R^n.$$

A quick calculation shows that the vector fields of Theorem 5 are

$$L = \{b, Ab, \cdots, A^{n-1}b, Ax\}$$
$$L_0 = \{b, Ab, \cdots, A^{n-1}b\}$$

and that

$$[L_0, b] = \{0\}.$$

Thus the reachable set at time t is

$$R(t) = \exp\{b, Ab, \cdots, A^{n-1}b\}e^{At}x_0.$$

Since $(b, Ab, \cdots, A^{n-1}b)$ represent constant vector fields, their exponentials correspond to translation. Thus this expression for $R(t)$ is equivalent to

$$R(t) = \{x : e^{At}x_0 + \eta, \qquad \eta \in \operatorname{span}(b, Ab, \cdots, A^{n-1}b)\}$$

which is a well-known result.

III. Input-Output Descriptions

One of the central ideas of system theory is the relationship between empirical "external" descriptions of systems and the various detailed "internal" models one can propose to account for the observed phenomena. In the case of linear systems this circle of ideas is expressed in terms of the relationship between the integral equation description

$$y(t) = \int_0^t \omega(t,\sigma)u(\sigma)\,d\sigma + \xi(t) \tag{6}$$

and the linear differential equation description

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \qquad y(t) = c(t)x(t).$$

In the present context, the internal descriptions will be by means of nonlinear differential equations. What substitute can be found for (6) as an external description? We can regard (6) as the first two terms in a power series expansion for y. The general series is the Volterra series:

$$y(t) = w_0(t) + \sum_{i=1}^{\infty} \int_0^t \int_0^{\sigma_1} \cdots \int_0^{\sigma_i} \omega_i(t, \sigma_1, \sigma_2, \cdots, \sigma_i)$$
$$\cdot u(\sigma_1)u(\sigma_2)\cdots u(\sigma_i)\,d\sigma_1 d\sigma_2 \cdots d\sigma_i.$$

This series, as it turns out, is just the right alternative for the integral equation (6) when dealing with linear analytic systems. This is due in part to the following theorem from [11].

Theorem 6: Suppose that $f(\cdot,\cdot): R^1 \times R^n \to R^n$ and $g(\cdot,\cdot): R^1 \times R^n \to R^n$ are continuous with respect to their first argument and analytic with respect to their second. Given any interval $[0, T]$ such that the solution of

$$\dot{x}(t) = f[t, x(t)], \quad x_0 = 0$$

exists on $[0, T]$, there exists an $\epsilon > 0$ and a Volterra series for

$$\dot{x}(t) = f[t, x(t)] + u(t)g[t, x(t)], \quad x(0) = x_0,$$
$$y(t) = h[x(t)] \quad (7)$$

with the Volterra series converging uniformly on $[0, T]$ to the solution of (7) provided $|u(t)| < \epsilon$. Moreover, the Volterra series is unique.

We give some indication of how the proof goes. For bilinear systems

$$\dot{x}(t) = A(t)x(t) + u(t)B(t)x(t), \quad x(0) = x_0, \quad y(t) = cx(t)$$

we can introduce $z(t)$ via $x(t) = \Phi_A(t, 0)z(t)$. Then $z(t)$ satisfies

$$\dot{z}(t) = u(t)\Phi_A^{-1}(t, 0)B(t)\Phi_A(t, 0)z(t) \overset{\text{def}}{=} u(t)\tilde{B}(t)z(t).$$

Using the Peano–Baker formula, we have

$$z(t) = \left[I + \int_0^t \tilde{B}(\sigma_1)\, d\sigma_1 + \int_0^t \int_0^{\sigma_1} \tilde{B}(\sigma_1)\tilde{B}(\sigma_2)u(\sigma_1)u(\sigma_2) \cdot d\sigma_2 d\sigma_1 + \cdots \right] x_0.$$

Then $cx(t)$ is characterized by the Volterra kernels

$$w_k(t, \sigma_1, \sigma_2 \cdots \sigma_k) = c\Phi_A(t, \sigma_1)B(\sigma_1)\Phi(\sigma_1, \sigma_2)B(\sigma_2) \cdots \Phi_A(\sigma_{k-1}, \sigma_k)x_0.$$

(see d'Alessandro *et al.* [12]). For time invariant bilinear systems, we have

$$w_n(t_1 \sigma_1 \cdots \sigma_n) = ce^{At}e^{-A\sigma_1}Be^{A\sigma_1}e^{-A\sigma_2}Be^{A\sigma_3} \cdots e^{-A\sigma_n}Be^{A\sigma_n}x_0.$$

An algorithm for constructing the Volterra series in the general case is given in [11]. It amounts to expanding all functions in their Taylor series, forming a sequence bilinear approximations of increasing accuracy, and computing the Volterra series for the bilinear approximations. The idea behind the derivation can be deduced from the following example.

Consider

$$\dot{x}(t) = -x(t) + x^2(t) + u(t)\,[1 + x^2(t)], \quad y(t) = x(t).$$

Now if we look at $x^2(t)$, it satisfies a certain differential equation which is linear in $u(t)$. More generally, x^p satisfies a differential equation which is also linear in u. We have

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ x^2(t) \\ x^3(t) \\ \vdots \end{bmatrix} = \begin{bmatrix} -1 & 2+u & 0 & \diagdown \\ 0 & -2 & 4+2u & \diagdown \\ 0 & 0 & -3 & \\ \diagdown & \diagdown & \diagdown & \end{bmatrix} \begin{bmatrix} x(t) \\ x^2(t) \\ x^3(t) \\ \vdots \end{bmatrix} + u \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix}$$

if $\int |u(\sigma)|\, d\sigma$ is small. Then we can expand this about $\int |u(\sigma)\, d\sigma = 0$ and get an expansion of $[x, \cdots, x^p]$ which is of order x^{p+1}. Thus we can expand x to any order by this "Carleman linearization" [13]. The approximate equations are all of the form $\dot{x}_p = A_p x_p + uB_p x_p + ub_p$, and the Volterra series for these bilinear systems can be computed using the Peano–Baker series.

As an example of Volterra series computation, we compute the frequency modulation Volterra series. The model is

$$\ddot{x}(t) + (a^2 + u(t))x(t) = 0, \quad x(0) = 0, \quad \dot{x}(0) = 1,$$
$$y(t) = x(t).$$

Writing this as a first-order system gives

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a^2 + u(t) & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix},$$
$$x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad y(t) = x_1(t)$$

and

$$y(t) = a^{-1} \sin at \int_0^t a^{-2} \sin a(t-\sigma) \sin a\sigma u(\sigma)\, d\sigma + \int_0^t \int_0^\sigma a^{-3} \sin a(t-\sigma) \sin a(\sigma-\rho) \cdot \sin a\rho u(\rho)u(\sigma)\, d\rho d\sigma + \cdots$$

with the nth kernel being

$$w_n = a^{-n} \sin a(t - \sigma_1) \sin a(\sigma_1 - \sigma_2) \cdots \sin a(\sigma_{n-1} - \sigma_n) \sin a\sigma_n.$$

As stated, the existence theorem for Volterra series is only a locally valid result. Global representation of a nonlinear system by means of an everywhere convergent Volterra series is not expected except for special cases. These special cases are the analogs of the entire functions of complex function theory—i.e., functions of a complex variable which have the whole complex plane as a domain of holomorphy. d'Alessandro *et al.* [12] have pointed out that, for bilinear systems, the Volterra series does converge for all locally bounded u. This is, perhaps, the first large class of nonlinear systems for which global convergence of the Volterra series has been established.

An obvious necessary condition for global convergence of the Volterra series is that there should be no finite escape times for the given initial condition of the system, regardless of the choice of u. Equivalently, one should ask that there be no finite escape time from any point reachable from x_0. As was pointed out before, a vector field on a manifold said to be complete if the solution through any initial point can be extended forwards and backwards in time to give a solution on $(-\infty, \infty)$. Thus one may reasonably restrict the discussion to complete vector fields. Even so, this is a problem of some subtlety in that it can happen that f and g are complete, and yet for

$$\dot{x}(t) = f[x(t)] + u(t)g[x(t)], \quad x(0) = x_0$$

there is a bounded function of time $u(\cdot)$ such that there exists finite escape time because $[f, g]$ is not complete. Thus in attempting to "globalize" Theorem 1, it is natural to ask that not only f and g be complete but that every vector field in $\{f, g\}_{LA}$ be complete. (In this connection it is appropriate to mention that if a finite number of complete vector fields generate a finite-dimensional Lie algebra, then every vector field in the algebra is complete.)

Suppose that $\{f, g\}_{LA}$ is a collection of complete analytic vector fields on a manifold M and suppose that every point on M can be reached from every other point on M (this assumption will be weakened later). If

$$y(t) = w_0(t) + \sum_{k=1}^{\infty} \int_0^t \int_0^{\sigma_1} \cdots \int_0^{\sigma_k} w_k(t, \sigma_1, \cdots, \sigma_k) \cdot u(\sigma_1) u(\sigma_2) \cdots u(\sigma_k) \, d\sigma_1 \, d\sigma_2 \cdots d\sigma_k$$

is the Volterra series for the linear-analytic system

$$\dot{x}(t) = f[x(t)] + u(t) g[x(t)], \qquad y(t) = h[x(t)], \qquad x(0) = x_0$$

and if for some $u(\cdot)$ defined on $[0, T]$ the Volterra series diverges, then we can still determine $y(t)$ from a knowledge of $u(\cdot)$ and the kernels w_k as follows.

From Theorem 6, given any bounded $u(\cdot)$, there exists ϵ, depending only on the bound on u, such that the Volterra series converges on $[0, \epsilon]$. Moreover, given any $u(\cdot)$ on $[0, T]$ there exists a trajectory on M generated by u. At each point on this trajectory, there is a Volterra series corresponding to the given differential equation and output map, but with altered initial state. At each point along the trajectory, there is an ϵ such that the Volterra series based on that starting state converges on $[0, \epsilon]$ for $u(t) \leqslant M$. It is not difficult to show (using the proof of [11, Theorem 1]) that the infimum of all these ϵ's is positive, say ϵ_0. Now we sum the Volterra series by splitting up the interval $[0, T]$ into subintervals of length ϵ_0 or less. Because of causality, i.e., because $y(t)$ does not depend on $u(\tau)$ for $\tau > t$, we can rearrange the Volterra series at time $n\epsilon_0$ as

$$y(t) = w_0^n(t) + \sum_{k=1}^{\infty} \int_{n\epsilon_0}^t \int_{n\epsilon_0}^{\sigma_1 + n\epsilon_0} \cdots \int_{n\epsilon_0}^{\sigma_k + n\epsilon_0} \cdot w_k^n(t, \sigma_1, \cdots, \sigma_k) u(\sigma_1 + n\epsilon_0) u(\sigma_2 + n\epsilon_0) \cdots u(\sigma_k + n\epsilon_0) \, d\sigma_1 \, d\sigma_2 \cdots d\sigma_k$$

where the altered Volterra kernels are obtained by integrating out the effect of u on the interval $[0, n\epsilon_0]$. At the same time, they are the kernels described by Theorem 6 with starting state $x(n\epsilon_0)$. Thus *the Volterra kernels at one point determine the Volterra kernels at all points which are reachable from that point*. It is not too difficult to see that if x is either reachable from x_0 or a point from which x_0 can be reached, then the Volterra series at x_0 determines the Volterra series at x. Extending this argument, we see that the Volterra series at x_0 determines the Volterra series at each point in $\{\exp \{f, g\}_{LA}\}_G x_0$.

We say that a Volterra series is *ϵ-summable* if, for every bounded $u(\cdot)$, there exists an $\epsilon > 0$ such that by splitting the time axis up into intervals of length ϵ or less and integrating out the kernels successively on $[0, \epsilon]$, $[\epsilon, 2\epsilon]$ $\cdots$, we can make the series convergent.

Theorem 7: Given a linear analytic system (7) with $\{f, g\}_{LA}$ a collection of complete vector fields, the Volterra series for starting state x_0 determines the Volterra series at any state in $\{\exp \{f, g\}_{LA}\}_G x_0$, and for any bounded u, the Volterra series is ϵ-summable to $y(\cdot)$.

An important aspect of this work is that the properties of the Lie algebra $\{f, g\}_{LA}$ mirror themselves in the properties of the Volterra series. We give an example of this which hinges on the following definition. A Lie algebra is *solvable* if the series of algebras $\mathcal{L}^k$ defined by $\mathcal{L}^1 = [\mathcal{L}, \mathcal{L}]$, $\mathcal{L}^{k+1} = [\mathcal{L}^k, \mathcal{L}^k]$ is the trivial algebra for some k. Checking for solvability involves computing derivatives and linear algebra. Solutions of differential equations are not required.

We say that a Volterra series is *finitely generated* if there exists a finite number of separable kernels $w_1, w_2, \cdots, w_r$ and an analytic function ϕ such that $y(t) = \phi(y_1(t), y_2(t), \cdots, y_r(t))$ with

$$y_i(t) = \int_0^t \cdots \int_0^{\sigma_{n_i}} w_i(t, \sigma_1, \cdots, \sigma_{n_i}) \cdot u(\sigma_1) \cdots u(\sigma_{n_i}) \, d\sigma_{n_i} \cdots d\sigma_1.$$

Naturally any Volterra series which is itself finite is finitely generated, but the converse is clearly false as $\exp \int_0^t u(\sigma) \, d\sigma$ demonstrates.

Theorem 8: Given (7), if the Lie algebra is solvable, then the Volterra series is finitely generated. Conversely, given any finitely generated Volterra series, there exists a realization of it with $\{f, g\}_{LA}$ solvable.

Proof (Sketch): The second statement comes from [11, Theorem 6] with minor changes. The first part is an immediate consequence of an analysis of Chen [14, Section 4]; we sketch how his argument goes. Let $\{f_i(x)\}_{i=1}^r$ be a basis for the Lie algebra generated by f and g. (This Lie algebra is finite dimensional since it is generated by a finite number of elements and is solvable.) We look for a solution of

$$\dot{x}(t) = f[x(t)] + u(t) g[x(t)], \qquad x(0) = x_0$$

which is of the form

$$x(t) = (\exp m_1(t) f_1)(\exp m_2(t) f_2) \cdots (\exp m_r(t) f_r) x_0.$$

Using the Baker–Campbell–Hausdorff formula, i.e., the vector field analog of the matrix identity

$$e^F B e^{-F} = B + [F, B] + \tfrac{1}{2} [F, B]] + \cdots$$

to expand $\exp(m_i(t) f) \exp(m_j(t) g) \exp(-m_i(t) f)$, we see that for small time, solutions of this form exist. Moreover, we can find the m_i as integrals of $u(\cdot)$ and exponential functions of integrals of u. (This part uses the solvability assumption! One can even order the basis so that this representation is valid for all time if the vector fields are complete.)

IV. The State-Space Isomorphism Theorem

There are two very important facts about modeling finite-dimensional linear I/O systems. The first asserts that any two models of the same I/O map are related by a change of basis, provided we assume that the systems are controllable and observable—thus making sure that the system has no internal parts which are irrelevant to the external description. The second asserts that any external description which can be realized by a finite dimensional linear system can be realized by one which is controllable and observable. In other settings, one of these results can be true without the other. For example, in the group theoretical setup of [15], the appropriate analog of the first is true but not the second. In the Hilbert space setup of [16], the second is true but not the first. In the present context, Sussmann [17] has shown that if a realization exists, a minimal realizations exists and that any two minimal realizations of a given I/O map are related by a smooth change of coordinates. In this section, we consider a special case of his theorems and some related results. For a well motivated discussion of these elegant results, see Sussman's contribution to [1].

Suppose that

$$\dot{x}(t) = a[x(t)] + u(t)\, b[x(t)], \quad y(t) = c[x(t)], \quad x(0) = x_0$$

and

$$\dot{z}(t) = f[z(t)] + u(t)\, g[z(t)], \quad y(t) = h[z(t)], \quad z(0) = z_0$$

are two linear-analytic realizations of the same I/O map. We would like to show that, under reasonable assumptions, there exist an analytic map ϕ such that $x = \phi(z)$ and $z = \phi^{-1}(x)$. If such a ϕ exists, then it follows that $\dot{x} = (\partial\phi)/(\partial z)\, \dot{z}$ and thus that

$$\frac{\partial \phi}{\partial z} f(\phi^{-1}(\cdot)) = a(\cdot)$$

$$\frac{\partial \phi}{\partial z} g(\phi^{-1}(\cdot)) = b(\cdot)$$

$$h[\phi^{-1}(\cdot)) = c(\cdot).$$

We say that a linear-analytic realization is *bilaterally controllable* on a manifold M if, given any two points x_1, x_2 on M, there exists $r \in \{\exp \{f, g\}_{LA}\}_G$ such that $rx_1 = x_2$. This is the same as asking that there be a sequence of points $x^1, x^2, \cdots, x^n$ with $x_1 = x^1$ and $x_2 = x^n$ and x^2 reachable from x^1 and x^3, x^4 reachable from x^3 and $x^5, \cdots$. That is, bilateral controllability means one can get from any point on M to any other point on M moving forward and backward in time. We say that a linear analytic realization on a manifold M is *observable* if, given any two distinct starting states x_1, x_2 in M, there exists an input $u(\cdot)$ whose choice may depend on x_1 and x_2, such that the output from x_1 differs from the output from x_2. Realizations which have these properties will be called *minimal*. Notice that the idea of dimension plays no role in this definition although one would hope that all minimal realizations of the same I/O map would be realized on manifolds of the same dimension, and this turns out to be the case.

Theorem 9 (Special case of Sussmann [17]): Suppose we are given two linear analytic systems

$$\dot{x}(t) = f[x(t)] + u(t)\, g[x(t)], \quad y(t) = h[x(t)], \quad x_0 = x(0)$$

and

$$\dot{z}(t) = a[z(t)] + u(t)\, b[z(t)], \quad y(t) = c[z(t)], \quad z_0 = z(0)$$

which are minimal realizations on manifolds M and N, respectively, of the same I/O map. Suppose that all the vector fields in $\{f, g\}_{LA}$ and $\{a, b\}_{LA}$ are complete. Then there exists a diffeomorphism $\phi : M \to N$ such that

$$\frac{\partial \phi}{\partial x} a[\phi^{-1}(x)] = f(x)$$

$$\frac{\partial \phi}{\partial x} b[\phi^{-1}(x)] = g(x)$$

$$c[\phi^{-1}(x)] = h(x).$$

Proof (Sketch): Since the realizations are minimal, there exists a map $\tilde{\phi} : M \to N$ which sends x_0 into z_0 and which sends any point x in M which is reachable from x_0 to the point z in N reached by the application of the same control sequence to the z system. Controllability insures that this map is onto N; it is one to one by observability. (All points are reachable by going forward and backward in time, so $\tilde{\phi}$ is defined on all of M and is onto N; no two starting states give identical I/O maps.) Clearly then, $\tilde{\phi}$ is a one to one and onto map. We have only to establish its smoothness properties. We want to show that $\tilde{\phi}$ is, in fact, analytic. Notice, however, that it must happen that for each $\{t_1, t_2, \cdots, t_n\}$, we have

$$\tilde{\phi}[(\exp t_1 f)(\exp t_2 g)(\exp t_3 f)(\cdots) x_0] = (\exp t_1 a)(\exp t_2 b)(\exp ta)(\cdots) x_0].$$

If we pick coordinates in M and coordinates in N, we can look at $\tilde{\phi}$ as a map of $R^n \to R^n$. The setup is diagramed in Fig. 4. The analyticity follows from the analyticity of the exp map and its inverse.

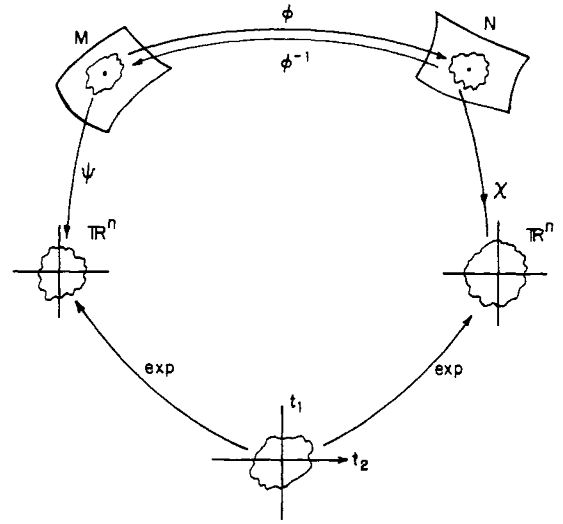

Fig. 4. Illustrating the proof of isomorphism theorem.

The other (more difficult) part of this circle of ideas is related to the existence of minimal realizations and is expressed by the following theorem.

Theorem 10: Given any I/O map which is realizable by an initialized linear-analytic system on a manifold M with $\{f, g\}_{LA}$ being complete, there exists a realization of the same I/O map on a manifold N such that the system is bilaterally controllable on N, and any two distinct states can be distinguished from the output for some input.

The proof of this theorem required a sharpening of the conditions of Theorem 2. In order to show that $\{\exp \{f, g\}_{LA}\}_G$ is always a differentiable manifold [18], [19] and also a generalization of the closed subgroup theorem of differential geometry [20]. Both these generalizations require new mathematical technique and yield new basic mathematics in return.

It is of interest to contrast Theorems 9 and 10 with the corresponding results for bilinear systems. Bilinear systems in R^n

$$\dot{x}(t) = Ax(t) + u(t)\, Bx(t), \quad y(t) = \langle c, x(t) \rangle, \quad x(0) = x_0$$

have a reachable set which is a proper subset of R^n. One says that they are minimal [12], [21] if there is no subspace of R^n which contains the reachable set for all time and if any two distinct starting states, x_1 and x_2, yield different I/O maps. The analog of Theorem 9 is that any bilinear system can be reduced to one which is minimal, and the analog of

Theorem 10 is that any two minimal bilinear systems are isomorphic in the following very strong, sense. If, A, B, c, and x_0 and F, G, h, and z_0 are the data which specify the systems, then there exists an invertible matrix P such that $PAP^{-1} = F$, $PBP^{-1} = G$, $cP'c = h$, and $Px_0 = z_0$. Thus in the case of bilinear systems, the structure one assumes to begin with is rather rigid, but the conclusion is also very precise (linear isomorphism). With linear-analytic systems the structure is less rigid and the conclusion is less precise.

We note in passing that the results we have given, together with a result of [12], allow us to give an answer to a question which has received some attention in the literature. That is, given a linear analytic systems, when does there exist a change of coordinates which puts it in bilinear form? The answer is the following. Compute the Volterra series for the stationary linear analytic system; if there exists a square matrix $T(t)$ with a rational Laplace transform, and vectors c and x_0, such that the kth kernel is

$$w_k(t, \sigma_1, \sigma_2, \cdots, \sigma_k) = c'T(t - \sigma_1)\, T(\sigma_1 - \sigma_2) \cdots T(\sigma_{k-1} - \sigma_k)\, x_0$$

then the Volterra series is bilinearly realizable (cf. [12]). If no such T, c, x_0 exists, then the system is not bilinearizable.

The work of Fleiss [22] is devoted to the derivation of some of the results on bilinear systems mentioned here using the Kleene–Schutzenberger representation theorem and the theory of formal power series.

The paper of Porter [23] contains further material developing Volterra series ideas in a different way.

V. Stochastic Equations

One of the most convincing applications of the controllability concept for linear systems occurs in the study of linear stochastic equations of the Ito type

$$dx(t) = A(t)\, x(t)\, dt + b(t)\, dw(t).$$

The density which results from a particular initial density can be expressed in terms of the density which results from $x(0) = 0$. This is

$$\rho(t, x) = \frac{1}{\sqrt{\det W(t)\, (2\pi)^n}} \exp\left[-\tfrac{1}{2} x' W^{-1}(t)\, x\right]$$

where W is a controllability Grammian,

$$W(t) = \int_0^t \Phi_A(t, \sigma)\, b(\sigma)\, b'(\sigma)\, \Phi_A'(t, \sigma)\, d\sigma$$

and W is invertible, as required for the expression for ρ to be meaningful, when the system has the controllability property. Thus controllability is exactly the right concept for deciding if smooth densities exist. Moreover, if A and b are constant and if the eigenvalues of A have negative real parts, then, under the controllability hypothesis, there is a unique invariant measure which is C^∞ (Gaussian actually) such that all measures approach it as t goes to infinity.

In fact, there is one more connection between controllability and the probability density which is even more striking. It is well known and easily seen [24] that the minimum value of

$$\eta(t_1, x_1) = \int_0^{t_1} u^2(t)\, dt$$

over the set of all u's which steer the deterministic system

$$\dot{x}(t) = A(t)\, x(t) + b(t)\, u(t)$$

from $x = 0$ at $t = 0$ to $x = x_1$ at $t = t_1$ is

$$\eta^*(x_1, t_1) = x_1' W^{-1}(t_1)\, x_1.$$

Comparing this with the formula for the density, we see that if we start at $x = 0$ at $t = 0$, then the value of the *probability density at any point is inversely proportional to the exponential of the "energy" required to get there.* In this section, we describe the extent to which these results carry over to linear analytic systems.

Because of the properties of the Ito calculus, stochastic equations on manifolds take a somewhat peculiar form. The basic well-posedness condition for the class of systems of interest here is expressed by the following theorem.

Theorem 11: Let $f, g: R^n \to R^n$ be analytic. Suppose that $M = \{x : x \in R^n,\ \phi_i(x) = 0,\ i = 1, 2, \cdots, p\}$ is an analytic variety in R^n. Given the Ito equation

$$dx = f(x)\, dt + g(x)\, dw, \qquad x(0) \in M \tag{8}$$

we can assert that $x(t)$ belongs to M with probability one, if for all x

$$G(x)\, \phi_i(x) = 0, \qquad i = 1, 2, \cdots, p$$
$$(F(x) - \tfrac{1}{2} G^2(x))\, \phi_i(x) = 0, \qquad i = 1, 2, \cdots, p$$

where F and G are given by

$$F(x) = f_1(x) \frac{\partial}{\partial x_1} + \cdots + f_n(x) \frac{\partial}{\partial x_n}$$
$$G(x) = g_1(x) \frac{\partial}{\partial x_1} + \cdots + g_n(x) \frac{\partial}{\partial x_n}.$$

Proof: (see [25] and [26].)

As application of this result we look at an example from [28]. Consider the Ito equation

$$dx = Axdt + Bxdw, \qquad x(0) = x_0, \qquad \|x_0\| = 1.$$

Under what circumstances can we assert that $\|x(t)\| = 1$ with probability one? In this case $\phi = x'x - 1$ and $G\phi(x) = 0$ if and only if B is skew symmetric. Likewise, $(F - (1/2)\, G^2)\, \phi(x) = 0$ if $A - (1/2)\, B^2$ is skew symmetric. Thus for example,

$$\begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix} = \begin{bmatrix} -1/2 & 1 \\ -1 & -1/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} dt + \begin{bmatrix} 0 & d\omega \\ -d\omega & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

defines a process which stays on the circle $\{x_1^2 + x_2^2 = \text{constant}\}$ almost surely in the mean-square sense. The apparent tendency of the drift term to pull the process inside the circle is offset by the Ito correction term.

Given a stochastic differential equation, the question arises as to whether or not smooth transition densities exist as they do in the linear controllable case. If they exist, then we denote by $\rho(t, x, y)$ the probability density at time t and point x given that at $t = 0$ the state was y.

The Fokker–Planck equation for the transition densities associated with the Ito equation (8) is easily expressed in terms of the above notation. It is

$$\left(\frac{\partial}{\partial t} + F(x) - \tfrac{1}{2} G^2(x)\right) \rho(t, x, y) = 0.$$

This equation has the formal appearance of a heat equation except that the second-order part, represented by the G^2

term, is not positive definite, so the classical existence theory does not apply. Fortunately, Hörmander [27] has worked out a theory of such equations which exactly fits the present case. Hörmander calls an operator

$$L = (F + G^2)$$

hypoelliptic if, for any given Schwartz distributions ϕ and ψ such that $L\phi(x) = \psi(x)$, it follows that ϕ is C^∞ off the support of the singular part of ψ after a suitable modification of ϕ on a zero measure. A necessary and sufficient condition for hypoellipticity is that $\{f, g\}_{LA}$ should span the tangent space of the manifold at each point. The $\partial/\partial t$ term in the Fokker–Planck equation must be combined with L to give an operator which is hypoelliptic on $M \times [0, \infty)$ where $[0, \infty)$ is the time axis. Since $\partial/\partial t$ spans the time direction, we have hypoellipticity if the smallest Lie algebra, which contains g and is closed under bracketing with f, spans the tangent space of M at each point of the manifold. These ideas were put together by Elliott [26], [27] to get the following results.

Theorem 11: Suppose that $\{f, g\}_{LA}$ consists of complete vector fields on a manifold M. Suppose that the smallest Lie algebra, which contains g and which is closed under bracketing with f, spans the tangent space of M at each point. Then the corresponding Ito equation

$$dx = \left(f + \frac{1}{2}\frac{\partial g}{\partial x} g\right) dt + g dw$$

defines C^∞ transition densities on M.

Notice that, in stating this theorem, we have taken the opposite point of view from Theorem 10 in that here f and g define vector fields on the manifold and the "correction term" $(1/2)(\partial g/\partial x) g$ is what is needed to insure that the given Ito equation evolves on the manifold. In Theorem 9, g and $f - (1/2)(\partial g/\partial x) g$ were the vector fields on the manifold.

Specific instances of processes meeting these conditions have been discussed in the literature. For example, in [28] the case of processes on spheres is studied in some detail.

Theorem 11 then asserts that some essential features of the linear stochastic differential equation can be transferred to manifolds. Only the last fact cited at the beginning of the section regarding the proportionality of the density to the exponential of the control energy has failed to be generalized. In special cases (essentially Abelian Lie groups), some progress has been made on finding a suitable extension of this result, but a general theory seems to require new developments.

There is a growing literature on the formulation and solution of filtering and stochastic control problems on manifolds (see, for example, [30]).

Appendix

The purpose of this appendix is to give the basic definition from the subject of differentiable manifolds in a form which is most compatable with the main body of this paper. We do so both to provide a convenient reference for the reader and also because there are many slight but potentially annoying differences in the general literature.

By a C^∞ *differentiable manifold*, we understand a triple (X, τ, Φ) whereby X is a set, τ is topology on X, and Φ is a set of continuous maps from open subsets of X into R^n with its usual topology. Both τ and Φ are subject to certain restrictions which we now describe. We ask that the topological space (X, τ) be a *Hausdorff space*, i.e., the open sets τ have the property that for any two distinct points x_1 and x_2 in X, there are elements of τ, $N(x_1)$ and $N(x_2)$ such that $N(x_1) \cap N(x_2)$ is empty with $x_1 \in N(x_1)$ and $x_2 \in N(x_2)$. We ask that (X, τ) be *separable*, that is, that there exist a countable subset of X which is dense in (X, τ). As a consequence of this, there is a countable subset of τ which generates all of τ under union; that is (X, τ) is *second countable*. So much for the topology. Now we consider the collection of maps $\Phi = \{\phi_i | \phi_i : M_i \to N_i\}$. We assume that there exists an integer n, a collection of open sets N_i, and continuous one to one maps $\phi_i : N_i \to M_i \subset R^n$ where M_i are open subsets of R^n such that $\phi_i^{-1} : M_i \to N_i$ are also continuous. This collection Φ is assumed to have the following additional properties.

(i) The domains of the ϕ_i form an open covering of X.
(ii) Each ϕ_i maps its domain bicontinuously onto an open subset of R^n.
(iii) If $N_i \cap N_j$ is not empty and if $\phi_i : N_i \cap N_j \to P_{ij}$ and $\phi_j : N_i \cap N_j \to Q_{ij}$, then $\phi_j \circ \phi_i^{-1}$ is an infinitely differentiable map of P_{ij} into Q_{ij}.
(iv) This collection Φ is maximal with respect to properties (ii) and (iii).

The locally defined maps ϕ_i are called *charts* and Φ is called, rather appropriately, an *atlas*. The last condition (iv) is not too important in that if we have some charts that we want to use, then we just declare Φ to be the collection of all charts which are compatible with the given ones. If in the previous paragraph we replace "infinitely differentiable" by "analytic", then we obtain an *analytic manifold*. (Incidentally, because we assume that the manifolds are second countable, they are automatically *paracompact*, a condition which is sometimes assumed in place of second countable.)

In addition to R^n itself, a standard source of manifolds are the so-called *differential (analytic) varieties*, in R^n described by the following theorem.

Theorem A: Let $\phi_1 \phi_2, \cdots, \phi_m$ be C^∞ (analytic) functions on R^n suppose that

$$\text{Rank } [(\nabla\phi_1, \nabla\phi_2, \cdots, \nabla\phi_n] = m$$

at each point on the set $M = \{x : \phi_i(x) = 0\}$. Then this set with the topology τ consisting of all sets of the form $0 \cap M$ 0 open in R^n can be given the structure of a C^∞ (analytic) differentiable manifold.

Proof: (see Singer and Thorpe [31, p. 120].)

A *submanifold* of a C^∞-manifold M is a pair (N, ϕ) where N is a C^∞-manifold and ϕ is a one to one mapping $\phi : N \to M$ such that $(d\phi)/(dx)$ has a trivial kernel at each point of N. In some cases, one defines an n-dimensional differentiable manifold M as a connected subset of the standard p-dimensional Cartesian space subject to the requirement that, at each point x of R^n which belongs to M, it is possible to establish a (generally curvilinear) coordinate system for R^p with basis elements $y_1, y_2, \cdots, y_p$ (R^p-valued C^∞-functions on R^p) such that in a neighborhood N of x, the intersection of M and N is exactly the set of points which satisfy

$$\begin{aligned} y_{n+1} &= 0 \\ y_{n+2} &= 0 \\ &\vdots \\ y_p &= 0. \end{aligned}$$

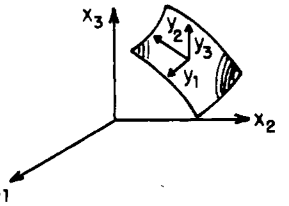

Fig. 5. Illustrating submanifold of R^3.

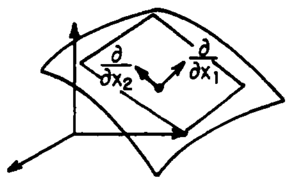

Fig. 6. Tangent space of M at m.

The remaining coordinate functions $y_1, y_2, \cdots, y_n$ then serve to define points in M unambiguously. Thus a differentiable manifold is a point set and a set of coordinate charts. This definition is fully consistent with the more abstract definition given above and, in fact, defines the differentiable structure alluded to in Theorem A (see Fig. 5).

Example: We may realize the n-dimensional sphere shell, as a submanifold of R^{n+1} as follows. Let ϕ_1 of Theorem A be $x_1^2 + x_2^2 \cdots x_{n+1}^2 - 1$. Then at each point of $M = \{x : \phi(x) = 0\}$ the rank of $\nabla\phi$ is one. Thus M can be given a manifold structure. Since $\nabla\phi$ has a trivial kernel, this makes (M, ϕ) a submanifold of R^{n+1}.

According to the Whitney imbedding theorem [32], any manifold is diffeomorphic to one which is imbedded in a Euclidean space.

We now set up the definition of a vector field on a manifold. Let M be a manifold and let $F^\circ(M)$ be the set of all C^∞ (analytic) functions on $M, F^\circ(M) = \{f | F: M \rightarrow R, \ f \in C^\infty$ or analytic$\}$. A tangent vector at $m \in M$ is an operator $v: F^\circ(M) \rightarrow R$ such that for all $f, g, \in F^\circ(M)$

(i) $v(f+g) = v(f) + v(g)$
(ii) $v(fg) = v(f) \cdot g(m) + v(g) f(m)$.

This definition does not look very geometrical, but observe first of all that the space of all tangent vectors at m is a real vector space, and second, if $x_1, x_2, \cdots, x_n$ defines a local coordinate system in the manifold, then for each i, $v(f) = \partial f)/(\partial x_i)]_m$ defines a tangent vector. In fact, one can see on the basis of a Taylor series argument that these vectors form a basis for the set of all tangent vectors at m, i.e., the *tangent space* at m. This vector space is denoted by $T_m(M)$ (see Fig. 6). By a vector field on manifold, we understand a map which assigns to each point of m an element of $T_m(M)$

$$m \longmapsto \sum_{i=1}^{n} a_i(x) \frac{\partial}{\partial x_i} \bigg| .$$

If the a_i are C^∞ (analytic), then this is called a C^∞ (analytic) *vector field*.

In R^n, we may then associate with every function $f: R^n \rightarrow R^n$ a vector field on R^n

$$f \sim f_1(x) \frac{\partial}{\partial x_1} + f_2 \frac{\partial}{\partial x_2} + \cdots f_n \frac{\partial}{\partial x_n} = F.$$

If (M, ϕ) is a submanifold of R^n such that $F\psi$ vanishes for all functions ψ which are constant on M, then in any local coordinate system of the type described in Theorem A we have

$$f_i \frac{\partial}{\partial x_i} = 0, \qquad i = 1, 2, \cdots, p.$$

Thus we can restrict F to $T(M)$ and get a vector field on M. The symbols $\partial/(\partial x_i)$ can be thought of as unit vectors to be compared with the symbols i, j, k, etc. in classical vector analysis. This notation associates with each vector field a partial differential operator whose domain is the differentiable functions on M. It is particularly useful, for example, in discussing the Fokker-Planck equation.

By the Whitney imbedding theorem, any manifold is diffeomorphic to a submanifold of R^n. In R^n, there are admissible coordinate systems which are global—e.g., the standard coordinate system for R^n. It also serves to identify points in any submanifold $M \subset R^n$ and thus we may describe vector fields on M in terms of this single global coordinate system.

When we speak of a differentiable dynamical system being on a manifold, what we mean is that at each point x_0 of M, we can pick local coordinates and express the evolution in a patch about x_0 by

$$\dot{x}(t) = a[x(t)] + u(t)\, b[x(t)].$$

However, it may happen that M is a subset of R^m, and we look only at the redundant coordinates

$$\dot{z}(t) = f[z(t)] + u(t)\, g[z(t)]$$

in R^n space. It is in this second way that one is most likely to encounter a practical problem.

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Linear Passive Networks: Functional Theory

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Abstract—**Linear passive time-variable networks are investigated primarily through the use of distributional kernels as applied to the scattering matrix treated in the time domain. Necessary and sufficient conditions for passivity are obtained, and the scattering matrix is shown to be a measure satisfying an energy form constraint. Lossless constraints pertinent to synthesis are developed while networks consisting of a finite number of circuit elements are considered in some detail. Examples illustrating interesting behavior are presented.**

I. Introduction

THE MATHEMATICAL FIELD of functional analysis is now recognized as a rich one with a varied, though modern, history [1]. Within functional analysis, the theory of distributions [2]-[4] plays a particularly interesting role, especially for physical systems. Indeed, motivation for the theory came, in part, from the circuit theory aspects of Heaviside's work [5] while more recent applications have led to the development of the properties of passive networks in a distributional framework, sometimes in the frequency domain [6, Sec. 3.5] and sometimes in the time domain [7]. Many of the results based upon distribution theory to date are summarized in the books of Doležal [8] and, more recently, Zemanian [9].

For time-invariant networks, there are also available more or less classical-type works [10]-[15] based upon frequency domain concepts. These concepts have been taken over to the time domain characterizations, primarily through the state [16], such that extensions to time-variable synthesis based upon the state [17], [18] become straightforward, albeit with strange results (as instabilities of passive structures [19]). Likewise, there are recent applicable developments in operator theory [20], especially with regard to resolution space concepts [21]-[24] as well as some distributional treatments of time-variable networks [25]-[28]. However, when one turns toward synthesis of time-variable networks, the functional analysis results are scarce [29]-[33]. Thus it seems that a functional analysis treatment of linear passive time-variable networks with an emphasis upon results important for synthesis is in order.

In this paper, some of the most important properties of linear passive networks are developed in terms of the time-varying scattering matrix [34] $\mathfrak{s}(t, \tau)$; this distributional kernel $\mathfrak{s}$ appears to be one of the most generally useful descriptions available, especially for synthesis. The paper is structured such that Sections IV and V contain the general results. Section II essentially serves as a review section where the underlying concepts of interest are defined; among these are a network and

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