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CONTROLLABILITY AND OBSERVABILITY OF POLYNOMIAL DYNAMICAL SYSTEMS*

JOHN BAILLIEUL

Scientific Systems, Inc., 54 Rindge Avenue Ext, Cambridge, MA 02140 U.S.A.
and
Department of Mathematics, Georgetown University, Washington, D.C. 20057, U.S.A.

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1. INTRODUCTION AND ALGEBRAIC PRELIMINARIES

THE IDEA of applying algebraic geometry to the study of control systems involving polynomial nonlinearities is fairly new. Some of the earliest work appeared in the 1976 paper of Sontag and Rouchaleau [7], on discrete time polynomial systems, and recently Sontag has reported a more extensive study of observability questions for this class of systems [8]. Applications to control systems evolving in continuous time have been studied by the present author and reported in [1]. The primary importance of this work is the use of the Hilbert basis theorem and elimination theory to significantly sharpen standard results on questions of integrability, stability, etc., for polynomial systems. This work is extended in the present paper, and for a wide class of systems we exploit the notions of finiteness in the Hilbert basis theorem to develop finitely verifiable conditions characterizing controllability and observability.

In the present section we shall recall a number of basic definitions from algebraic geometry which will be used throughout the remainder of the paper. For additional details the reader is referred to any standard text such as [3] or [6]. Also, some simple ideas from differential geometry and Lie theory will be used in the next section. Standard texts covering this material are [5] and [12].

We let $k = \mathbb{C}$ or $\mathbb{R}$. For each positive integer p and n-tuple x of elements in k, let $x^{[p]}$ denote the $\binom{n+p-1}{p}$-tuple of weighted p-forms in the components of x (i.e., $x^{[p]} = {}^t(x_1^p, \alpha_1 x_1^{p-1} x_2, \alpha_2 x_2^{p-1} x_3, \ldots, x_n^p)$†), where the entries are ordered lexicographically, and the weights are chosen so that $\|x^{[p]}\| = \|x\|^p$. Hence a typical entry is of the form $\alpha x_1^{p_1} x_2^{p_2} \ldots x_n^{p_n}$ where $\sum p_i = p$ and $\alpha = \sqrt{p!/(p_1! \ldots p_n!)}$. We define $x^{[0]}$ to be the scalar 1 for all n-tuples x.

Throughout this paper we shall think of elements of k^n as column vectors. $k[s_1, \ldots, s_n]$ is the ring of polynomials in the indeterminates $s_1, \ldots, s_n$; this notation will frequently be abbreviated $k[s]$. An *algebraic set* in k^n is the set of zeros for some subset of polynomials in $k[s]$. Thus if $\mathcal{Q} \subseteq k[s]$, we have an associated algebraic set $V(\mathcal{Q}) = \{x \in k^n : f(x) = 0 \text{ for all } f \in \mathcal{Q}\}$. Let $\mathcal{V}_{\mathcal{Q}}$

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† For any matrix A, ${}^tA$ denotes the transpose of A.

denote the smallest ideal in $k[s]$ which contains $\mathcal{Q}$. Then $V(\mathcal{Q}) = V(\mathscr{V}_{\mathcal{Q}})$. Dually, if $S \subseteq k^n$, define the ideal $\mathscr{V}(S) = \{f \in k[s] : f(x) = 0 \text{ for all } x \in S\}$. Then $S \subseteq V(\mathscr{V}(S))$. If $\mathscr{V}$ is any ideal in $k[s]$ we have $\mathscr{V} \subseteq \mathscr{V}(V(\mathscr{V}))$. $\mathscr{V}(V(\mathscr{V}))$ is called the *radical* of $\mathscr{V}$ and it is the largest ideal defining $V(\mathscr{V})$ in the sense that it contains as a subset any other defining ideal.

If $f \in k[s]$, $x \in k^n$ we define the *differential of f at x* to be the linear function $d_x f: k^n \to k$ given by $d_x f(v) = \sum_{i=1}^{n} (\partial f/\partial s_i)(x)v_i$. If $f \in k[s]$ and $F(s)$ is a column vector whose entries are elements of $k[s]$, we define the *Lie derivative* of f with respect to F by $L_F f(s) = d_s f(F(s)) = \sum(\partial f/\partial s_i)(s)F_i(s)$. Next, given a set $\mathcal{Q} \subseteq k[s]$ and a set $\mathscr{T}$ whose elements are column vectors of polynomials, let $I(\mathcal{Q}; \mathscr{T})$ denote the smallest polynomial ideal in $k[s]$ which contains $\mathcal{Q}$ and is closed under Lie differentiation with respect to elements of $\mathscr{T}$.

Let V be an algebraic set defined by the equations $f_1(x) = \ldots = f_r(x) = 0$. The *tangent space to V at x*, denoted $T_x V$, is the vector space of all $v \in k^n$ such that $d_x f_i(v) = 0$ for $i = 1, \ldots, r$. An algebraic set V is said to be *irreducible* if there do not exist algebraic sets V_1 and V_2 such that $V = V_1 \cup V_2$ with $V \neq V_i$. Irreducible algebraic sets are also called *algebraic varieties*. Any algebraic set is the union of finitely many algebraic varieties. If V is an algebraic variety and $x \in V$ is such that $\dim T_x V = \min \dim T_y V$, x is called a *simple point*. All other points in the variety V are called *singular points*.

The *geometric dimension* of an algebraic variety is the dimension of the tangent space at any simple point. It is usual to define the *algebraic dimension* of an algebraic variety as the transcendence degree of the field of fractions of the coordinate ring. The algebraic dimension will coincide with the geometric dimension when $k = \mathbb{C}$ but if $k = \mathbb{R}$ the algebraic dimension only provides an upper bound on the geometric dimension. The dimension of a (reducible) algebraic set is the maximum of the dimensions of its irreducible components.

Let $V \subset k^n$ and $W \subset k^m$ be arbitrary varieties. An *algebraic mapping* $f: V \to W$ is a mapping of the form $f(x_1, \ldots, x_n) = (f_1(x), \ldots, f_m(x))$ where $f_i \in k[s]$ and such that whenever $g \in \mathscr{V}(W)$ it follows that $g \circ f \in \mathscr{V}(V)$. If $f: V \to W$ is an algebraic mapping it is immediate from the definition that for each $y \in W$ the fiber $f^{-1}(y)$ is a closed algebraic subset of V. As y ranges over W, the dimension of the algebraic sets $f^{-1}(y)$ remains constant on a non-empty open subset of W. In general, however, there will be points in W where the dimension jumps. For certain algebraic mappings of k^n into k^m the following theorem gives a bound on the dimension of the fibers. This result will be useful in Section 3 in connection with observability.

THEOREM 1.1. Let $f = (f_1, \ldots, f_m): k^n \to k^m$ be a mapping such that each f_i is a homogeneous polynomial with $\deg f_i \geqslant 1$. For each $y \in k^m$, $f^{-1}(y)$ is an algebraic set in k^n and $\dim f^{-1}(y) \leqslant \dim f^{-1}(0)$. (Here "dimension" means "algebraic dimension".)

The proof of this theorem is somewhat involved and is omitted. It may be found in [1].

2. CONTROLLABILITY

We wish to study systems of nonlinear controlled differential equations of the form

$$\dot{x}(t) = f(x(t)) + u(t)g(x(t)) \tag{1}$$

where f and g are vector fields with polynomial dependence on their arguments. It is usual to regard $x(t)$ as belonging to $\mathbb{R}^n$ for each $t \geqslant 0$, but we shall find it useful to assume more generally that $x(t) \in k^n$ where $k = \mathbb{R}$ or $\mathbb{C}$. $u(.)$ is a k-valued piecewise-C^∞ function. We are thus restricting our attention to scalar controls. This is done only to simplify the exposition, and the extension of all our results to the case of vector controls is straightforward.

Definition 2.1. (i) Given $x \in k^n$, the set of all $y \in k^n$ such that there is a $t > 0$, a piecewise-C^∞ control $u(.)$ and a trajectory $x(.)$ defined by (1) with $x(0) = x$ and $x(t) = y$ is called the *reachable set from x at time t*. Denote this set by $\mathscr{A}(x, t)$.

(ii) The set $\mathscr{A}(x) = \bigcup_{t \geqslant 0} \mathscr{A}(x, t)$ is called the *reachable set from x*.

THEOREM 2.1. Let V be an algebraic set in k^n. If $\mathscr{A}(x_0) \subseteq V$ for each $x_0 \in V$ then $I(\mathscr{V}(V); \{f, g\}) = \mathscr{V}(V)$. If for any ideal $\mathscr{V}$ defining V $I(\mathscr{V}, \{f, g\}) = \mathscr{V}$ then $\mathscr{A}(x_0) \subseteq V$ for each $x_0 \in V$.

This theorem is proved in [1]. It is pointed out that the statement "$\mathscr{A}(x_0) \subseteq V$ for each $x_0 \in V$" does not imply "$I(\mathscr{V}; \{f, g\}) = \mathscr{V}$" for an arbitrary $\mathscr{V}$ defining V. Let $\mathscr{V}$ be the ideal in $k[s_1, s_2]$ generated by the polynomials $\phi_1(s_1, s_2) = s_1^2$ and $\phi_2(s_1, s_2) = s_2$ and let $f(s_1, s_2) = \binom{0}{s_1}$, $g(s_1, s_2) = \binom{0}{0}$. $V(\mathscr{V}) = \{\binom{0}{0}\}$ and it is trivial that $\mathscr{A}(x_0) \subseteq V$ for each $x_0 \in V$. Nevertheless, a simple calculation shows that $I(\mathscr{V}; \{f, g\}) = \mathscr{V}(V)$ and $\mathscr{V}(V)$ properly contains $\mathscr{V}$.

Definition 2.2. The system (1) is said to have the *strong accessibility property* on k^n if for all $x \in k^n$ and all $t > 0$, $\mathscr{A}(x, t)$ has non-empty interior (in the standard k^n topology).

Example 2.1. Consider the linear system

$$\dot{x}(t) = Ax(t) + bu(t).$$

It is well known that if rank $(b, Ab, \ldots, A^{n-1}b) = n$, then for all $x \in k^n$ and $t > 0$, $\mathscr{A}(x, t) = k^n$. On the other hand, if rank $(b, Ab, \ldots, A^{n-1}b) < n$ there is a $v \in k^n$ such that ${}^t v \,.\, A^k \,.\, b = 0$ for $k = 0, 1, \ldots, n-1$. Let $\mathscr{V} = I(\{{}^t v \,.\, s\}, \{As\})$, and let $V = V(\mathscr{V})$. It is easy to see that $\mathscr{V}(V)$ is closed under Lie differentiation with respect to As and b, so that by Theorem 2.1 the motion of the system can be confined to a proper (linear) subvariety of k^n. Thus, for linear systems the notion of strong accessibility is the same as the usual notion of controllability.

Example 2.2. Consider the system

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} x_2(t)^2 \\ -x_1(t)x_2(t) \end{pmatrix} + \begin{pmatrix} 0 \\ u(t) \end{pmatrix}$$

in $\mathbb{R}^2$. We shall subsequently see that this system has the strong accessibility property on $\mathbb{R}^2$. For any initial point $x \in \mathbb{R}^2$, however, the reachable set $\mathscr{A}(x)$ is not all of $\mathbb{R}^2$. Indeed, it is immediate that $\mathscr{A}(x)$ lies in the half-plane $x_1 \geqslant x_1(0)$.

We shall now present a criterion for a system of the form

$$\dot{x}(t) = Ax(t)^{[p]} + bu(t) \tag{2}$$

to have the strong accessibility property. This criterion will be analogous to the usual rank condition for linear systems (Example 2.1). If we write $f(x) = Ax^{[p]}$, then the pth differential $d^p f$

defines a symmetric p-linear mapping $k^n \times k^n \times \ldots \times k^n \to k^n$. We wish to consider a set B of vectors defined recursively in terms of A and b as follows: $b \in B$ and whenever $v_1, \ldots, v_p \in B$ it follows that $d^pf(v_1, \ldots, v_p) \in B$. Also, define the *order* or element of B recursively by saying the order of b is 1 and the order of $d^pf(v_1, \ldots, v_p)$ is the sum of the orders of $v_1, \ldots, v_p$ plus 1. The connection with controllability is given by the following.

THEOREM 2.2. The system (2) has the strong accessibility property on k^n if and only if the elements of B of order $\leqslant 1 + p + \ldots + p^{n-1}$ span k^n.

The proof of this theorem will be by means of two lemmas.

LEMMA 2.1. The system (2) has the strong accessibility property on k^n if and only if B spans k^n.

Proof. First, suppose B spans k^n. Let g denote the Lie algebra of vector fields generated by $Ax^{[p]}$ and b; i.e., g is the smallest Lie algebra of analytic vector fields defined on k^n which contains both $Ax^{[p]}$ and b. Let $\mathscr{S}_0$ denote the ideal (in the sense of Lie theory) in g generated by the vector field b. Finally, for each $x \in k^n$ let $\mathscr{S}_0(x)$ denote the set of vector fields in $\mathscr{S}_0$ evaluated at x. From the work of Sussman and Jurdjevic [9], we know that if $\dim \mathscr{S}_0(x) = n$ for each $x \in k^n$, then (2) has the strong accessibility property on k^n. Therefore, since $B \subseteq \mathscr{S}_0(x)$ and we assumed B spans k^n it follows that the system (2) has the strong accessibility property on k^n.

Suppose, on the other hand, that B spans only a subspace of k^n. Then there is some nonzero vector $v \in k^n$ such that ${}^tv \cdot w = 0$ for all $w \in B$. Let $\mathscr{Q}$ consist of the single homogeneous polynomial ${}^tv \cdot s = v_1s_1 + v_2s_2 + \ldots + v_ns_n$, and let $\mathscr{V} = I(\mathscr{Q}; \{As^{[p]}, b\})$. Since $\mathscr{V} = I(\mathscr{V}; \{As^{[p]}, b\})$ it follows from Theorem 2.1 that for each $x_0 \in V(\mathscr{V})$, $\mathscr{A}(x_0) \subseteq V(\mathscr{V})$. Since $\mathscr{V}$ contains homogeneous polynomials of degree $\geqslant 1$, $V(\mathscr{V})$ is a proper closed algebraic subset of k^n. This implies for $x_0 \in V(\mathscr{V})$ $\mathscr{A}(x_0)$ has empty interior in the topology of k^n. Thus (2) does not have the strong accessibility property on k^n.

LEMMA 2.2. Each element in the set B is a linear combination of elements whose orders do not exceed $1 + p + \ldots + p^{n-1}$.

Proof. We shall show that any element in B of order greater than $1 + p + \ldots + p^{n-1}$ can be written as a linear combination of elements of strictly lower order. By applying this argument sufficiently many times we obtain the conclusion of the lemma.

Call an element of B *primitive* if it cannot be expressed as a linear combination of elements of strictly lower order. Suppose $v \in B$ is a primitive element of order greater than $1 + p + \ldots + p^{n-1}$. Then we may write $v = d^pf(v_1^1, v_2^1, \ldots, v_p^1)$ where each v_i^1 is primitive. (If some v_i^1 were not primitive, v itself could not be primitive.) Also, at least one of the v_i^1, say v_1^1, is of order greater than $1 + p + \ldots + p^{n-2}$. Now we may write $v_1^1 = d^pf(v_1^2, \ldots, v_p^2)$ where each v_i^2 is primitive and at least one of the v_i^2, say v_1^2, is of order greater than $1 + p + \ldots + p^{n-3}$. Continuing in this way, we produce a string of elements $v, v_1^1, v_1^2, \ldots, v_1^{n-1}$ in B which are all primitive and such that the order of v_1^k is greater than $1 + \ldots + p^{n-k-1}$. Let $v_1^0 = v$ and $v_1^n = b$. Then either $v_1^0, v_1^1, \ldots, v_1^n$ is a set of $n + 1$ linearly independent elements in R^n or else there is a nontrivial linear combination $\sum_{i=0}^{n} \alpha_i v_1^i = 0$. The former conclusion is obviously impossible while the latter violates the condition that each v_1^i $(0 \leqslant i \leqslant n)$ is primitive. This forces us to abandon our assumption that there exists a primitive element of order greater than $1 + \ldots + p^{n-1}$, and this proves Lemma 2.2.

Proof of Theorem 2.2. From Lemma 2.2 B spans k^n if and only if the elements in B of order $\leqslant 1 + p + \ldots + p^{n-1}$ span k^n. Theorem 2.2 is therefore a direct consequence of Lemma 2.1.

3. OBSERVABILITY

In this section we wish to study systems with output:

$$\dot{x}(t) = Ax(t)^{[p]} + u(t)Bx(t)^{[q]}, \tag{3}$$

$$y(t) = Cx(t)^{[r]}, \tag{4}$$

where $x(t) \in k^n$, $y(t) \in k$ and A, B and C are matrices of the appropriate dimensions with entries in k. Assume $p, r \geqslant 1$ and $q \geqslant 0$. Also assume that the vector field $Ax^{[p]}$ is complete in the usual sense that corresponding integral curves are defined for all times $t \in (-\infty, \infty)$. In (3) and (4) we are restricting our attention to scalar output as well as scalar input. Again, we remark that the extension to the case of vector inputs and outputs makes the notation cumbersome, but it presents no essential difficulty.

Some additional notation is unavoidable at this point. For any piecewise-C^∞ control $u(.)$ let $\gamma_t^u(x)$ denote the flow determined by the differential equation (3). For simplicity, let $f(x) = Ax^{[p]}$, $g(x) = Bx^{[q]}$ and $h(x) = Cx^{[r]}$. For each $x \in k^n$ let $F(x)$ denote the set of all $y \in k^n$ such that for each $\delta > 0$, $h(\gamma_t^0(x)) \equiv h((\gamma_t^0(y))$ on $[0, \delta]$ where $\gamma_t^0(.)$ is the flow generated by the input $u \equiv 0$. In other words, $F(x)$ is the set of all initial states for the system (3) which cannot be distinguished from the initial state x using the zero input and observing the output $y(t)$ on any interval $[0, \delta]$. It is easy to see that the relation of state x being indistinguishable from state y in this sense is an equivalence relation.

Next define the set of polynomials $\mathscr{F} = \{L_f^k h\colon k = 0, 1, \ldots\}$. This is related to F by the following.

THEOREM 3.1. $F(x) = F(y)$ if and only if $\phi(x) = \phi(y)$ for all $\phi \in \mathscr{F}$.

Proof. Suppose $F(x) = F(y)$ (i.e., $x \in F(y)$ or equivalently $y \in F(x)$). Then $h(\gamma_t^0(x)) \equiv h(\gamma_t^0(y))$ on $[0, \delta]$ for every $\delta > 0$. Differentiating both sides k times yields $L_f^k h(\gamma_t^0(x)) \equiv L_f^k h(\gamma_t^0(y))$. Hence $L_f^k h(x) = L_f^k h(y)$ for $k = 0, 1, \ldots$ and we have proved the "only if" portion of the theorem.

Suppose, on the other hand, that $\phi(x) = \phi(y)$ for all $\phi \in \mathscr{F}$. Let $\delta > 0$ be given. We wish to show that $h(\gamma_t^0(x)) = h(\gamma_t^0(y))$ for each $t \in [0, \delta]$. We know that $h(\gamma_t^0(x))$ and $h(\gamma_t^0(y))$ are analytic functions of t. Expanding these about $t = 0$ we obtain

$$h(\gamma_t^0(y)) = h(y) + L_f h(y)t + \tfrac{1}{2}L_f^2 h(y)t^2 + \ldots$$

and

$$h(\gamma_t^0(x)) = h(x) + L_f h(x)t + \tfrac{1}{2}L_f^2 h(x)t^2 + \ldots.$$

Since by hypothesis the Taylor coefficients are equal, $h(\gamma_t^0(x)) = h(\gamma_t^0(y))$ on their common interval of convergence, say $(-\delta_1, \delta_1)$. A straightforward analytic continuation argument shows that indeed $h(\gamma_t^0(x)) = h(\gamma_t^0(y))$ for all $t \in [0, \delta]$. This proves the theorem.

By observing the output $y(t)$ corresponding to a certain nonzero input $u(t)$ it may be possible to distinguish two initial states x and y even though $F(x) = F(y)$. It is useful to define $G(x)$ to be the set of all $y \in k^n$ such that for each $\delta > 0$, $h(\gamma_t^u(x)) \equiv h(\gamma_t^u(y))$ on $[0, \delta]$ for every choice of piecewise-C^∞ control u. In other words, $G(x)$ is the set of all initial states for the system (3) which for *any* piecewise-C^∞ input u are indistinguishable (by observations of the output y). Again one notes that indistinguishability in this sense is an equivalence relation. The equivalence classes of the "F" relation form a coarser partition of k^n in the sense that for each $x \in k^n$, $G(x) \subseteq F(x)$. In general this containment is strict, but for the special case of linear systems (where $p = r = 1$, $q = 0$) the reverse inclusion also holds.

Define the following sets of polynomials. $\mathscr{G}_0 = \{h\}$, $\mathscr{G}_k = \{L_f\phi, L_g\phi : \phi \in \mathscr{G}_{k-1}\}$. Let $\mathscr{G} = \bigcup_{k \geq 0} \mathscr{G}_k$. The following theorem is proved by Hermann and Krener [4].

THEOREM 3.2. If $G(x) = G(y)$ then $\phi(x) = \phi(x)$ for all $\phi \in \mathscr{G}$.

Example 3.1. Consider the system in $\mathbb{R}^2$

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} x_2(t) \\ -x_1(t) \end{pmatrix} + \begin{pmatrix} u(t) \\ 0 \end{pmatrix},$$
$$y(t) = x_1(t)^2.$$

Under the zero input the motion of the system for any nonzero initial state is confined to a circle centred at the origin. We can list the first few elements of $\mathscr{F}$. $h(s_1, s_2) = s_1^2$, $L_f h(s_1, s_2) = 2s_1s_2$, $L_f^2 h(s_1, s_2) = 2s_2^2 - 2s_1^2$, $L_f^3 h(s_1, s_2) = -8s_1s_2$. Apparently $\mathscr{F}$ consists of various multiples of s_1^2, s_1s_1 and $s_2^2 - s_1^2$. It follows from Theorem 3.1 that $F(x) = \{x, -x\}$ for each $x \in \mathbb{R}^2$. On the other hand, $L_g h(s_1, s_2) = 2s_1$ and $L_f L_g h(s_1, s_2) = 2s_2$ are elements of $\mathscr{G}$. Hence $G(x) = \{x\}$ for all $x \in \mathbb{R}^2$. This provides an example of a simple nonlinear system for which the output distinguishes among all initial states, but for which output from the free motion alone (i.e., zero input) fails to distinguish certain states.

The sets $F(0)$ and $G(0)$ are of special practical importance since it is often important in applications to know whether the system is at rest or not. The remaining results of this section show that these sets are also of special theoretical interest. Let $\hat{\mathscr{G}}_k = \bigcup_{j=0}^{k} \{\phi \in \mathscr{G}_j : \deg \phi \geq 1\}$ and let $\hat{\mathscr{G}} = \bigcup_{k \geq 0} \hat{\mathscr{G}}_k$. $\mathscr{F}$ and $\hat{\mathscr{G}}$ determine algebraic sets, the importance of which is shown by the following.

THEOREM 3.3. (i) $F(0) = V(\mathscr{F})$. (ii) $G(0) \subseteq V(\hat{\mathscr{G}})$.

Proof. Let $x \in F(0)$; then by Theorem 3.1 $\phi(x) = \phi(0) = 0$ for all $\phi \in \mathscr{F}$. Hence $x \in V(\mathscr{F})$. On the other hand if $x \in V(\mathscr{F})$, then $\phi(x) = 0$ for all $\phi \in \mathscr{F}$. Hence $\phi(x) = \phi(0)$ for all $\phi \in \mathscr{F}$ and it follows from Theorem 3.1 that $x \in F(0)$. This proves (i), (ii) is proved similarly using Theorem 3.2.

$\mathscr{F}$ and $\hat{\mathscr{G}}$ determine ideals $\mathscr{V}_{\mathscr{F}}$ and $\mathscr{V}_{\hat{\mathscr{G}}}$ in $k[s]$. Moreover, it is a direct consequence of the Hilbert basis theorem that for some positive integers k_1, k_2 we may take $\mathscr{F}_{k_1} = \{h, L_f h, \ldots, L_f^{k_1} h\}$ as a basis for $\mathscr{V}_{\mathscr{F}}$ and $\hat{\mathscr{G}}_{k_2}$ (defined above) as a basis for $\mathscr{V}_{\hat{\mathscr{G}}}$. Hence $V(\mathscr{F}) = V(\mathscr{V}_{\mathscr{F}}) = V(\mathscr{F}_{k_1})$ and $V(\hat{\mathscr{G}}) = V(\mathscr{V}_{\hat{\mathscr{G}}}) = V(\hat{\mathscr{G}}_{k_2})$. The main problem is to decide when $F(0) = \{0\}$ (resp. $G(0) = \{0\}$). In light of Theorem 3.3 this will be the case if $V(\mathscr{F}) = \{0\}$ (resp. $V(\hat{\mathscr{G}}) = \{0\}$). We may find $V(\mathscr{F})$ and $V(\hat{\mathscr{G}})$ explicitly by solving the systems of simultaneous homogeneous algebraic equations

$\phi(x) = 0$ for $\phi \in \mathscr{F}_{k_1}$ and $\psi(x) = 0$ for $\psi \in \hat{\mathscr{G}}_{k_2}$, respectively. These calculations can be carried out using the techniques of elimination theory as outlined in [10] and [11]. In particular $V(\mathscr{F}) = \{0\}$ (resp. $V(\hat{\mathscr{G}}) = 0$) if and only if the *inertia forms* for the corresponding systems of algebraic equations do not vanish. (The inertia forms for a system of simultaneous homogeneous algebraic equations are defined in Sections 80 and 81 or [10]. These represent a generalization of the determinant for a system of n linear equations in n unknowns. They are homogeneous polynomials in the coefficients of the equations, and they all vanish if and only if the system has a nontrivial solution.)

The non-vanishing of the inertia forms is a finitely verifiable sufficient condition for $F(0) = \{0\}$ and $G(0) = \{0\}$ respectively. We shall now see that this same condition is sufficient for the sets $F(x)$ and $G(x)$ to have finite cardinalities for each $x \in k^n$.

Let $x \in k^n$ be arbitrary but fixed and for each $\phi \in \mathscr{F}$ let $a_{\phi,x} \in k$ be defined by $a_{\phi,x} = \phi(x)$. For each $\psi \in \hat{\mathscr{G}}$ let $b_{\psi,x} \in k$ be defined by $b_{\psi,x} = \psi(x)$. By Theorem 3.1, $F(x) = \{y \in k^n : \phi(y) - a_{\phi,x} = 0$ for all $\phi \in \mathscr{F}\}$, and this is a subset of $\bar{F}(x) = \{y \in k^n : \phi(y) - a_{\phi,x} = 0$ for all $\phi \in \mathscr{F}_{k_1}\}$. Similarly, $G(x)$ is a subset of $\bar{G}(x) = \{y \in k^n : \psi(y) - b_{\psi,x} = 0$ for $\psi \in \hat{\mathscr{G}}_{k_2}\}$. For each $x \in k^n$, $\bar{F}(x)$ and $\bar{G}(x)$ are fibers of homogeneous polynomial mappings. Theorem 1.1 applies to show that $\dim \bar{F}(x) \leqslant \dim \bar{F}(0)$ and $\dim \bar{G}(x) \leqslant \dim \bar{G}(0)$.

THEOREM 3.4. If the inertia forms for the system of simultaneous homogeneous algebraic equations $\phi(x) = 0$ for $\phi \in \mathscr{F}_{k_1}$ (resp. $\psi(x) = 0$ for $\psi \in \hat{\mathscr{G}}_{k_2}$) do not vanish, or equivalently, if the only solution to the system $\phi(x) = 0$ for $\phi \in \mathscr{F}_{k_1}$ (resp. $\psi(x) = 0$ for $\psi \in \hat{\mathscr{G}}_{k_2}$) is $x = 0$, then $F(x)$ (resp. $G(x)$) is a finite set for each $x \in k^n$, and $F(0) = \{0\}$ (resp. $G(0) = \{0\}$).

Proof. Suppose the only solution to the system $\phi(x) = 0$ for $\phi \in \mathscr{F}_{k_1}$ is $x = 0$. This implies $\bar{F}(0) = \{0\}$ (and *a fortiori* $F(0) = \{0\}$). Thus $\dim \bar{F}(0) = 0$ and as noted above $\dim \bar{F}(x)$ must also be zero for each $x \in k^n$. But the dimension of an algebraic set is zero if and only if that set is finite. Since $F(x) \subseteq \bar{F}(x)$, $F(x)$ must also be finite, proving the theorem for F. The same argument is valid with F, $\bar{F}$ and $\mathscr{F}_{k_1}$ replaced by G, $\bar{G}$ and $\hat{\mathscr{G}}_{k_2}$.

COROLLARY 3.1. The inertia forms for the system of simultaneous homogeneous algebraic equations $\phi(x) = 0$ for $\phi \in \mathscr{F}_{k_1}$ do not vanish if and only if $F(x)$ is a finite set for each $x \in k^n$ with $F(0) = \{0\}$.

Proof. The "only if" portion is the statement of Theorem 3.4. Suppose $F(x)$ is finite for each $x \in k^n$ with $F(0) = \{0\}$. From Theorem 3.3 and the definition of $\mathscr{F}_{k_1}$ we see that $V(\mathscr{F}_{k_1}) = \{0\}$, which means the only solution to the system of simultaneous homogeneous algebraic equations $\phi(x) = 0$ for $\phi \in \mathscr{F}_{k_1}$ is $x = 0$. From the results in Section 80 in [10] this implies the inertia forms are zero.

Remark. An open question is whether $G(0) = \{0\}$ implies the vanishing of the inertia forms for the system $\psi(x) = 0$ for $\psi \in \hat{\mathscr{G}}_{k_2}$.

Example 3.2. Consider the linear system

$$\dot{x} = Ax + bu,$$

$$y = cx.$$

Here $\mathscr{F} = \hat{\mathscr{G}} = \{cs, cAs, \ldots, cA^k s, \ldots\}$. We know $\mathscr{V}_{\mathscr{F}}$ is generated by $\{cs, cAs, \ldots, cA^{n-1}s\}$ (the Cayley–Hamilton theorem!). There is one inertia form for the system $cA^k x = 0, k = 0, 1, \ldots, n-1$; it is the determinant of the matrix

$$\begin{pmatrix} c \\ cA \\ \cdot \\ \cdot \\ cA^{n-1} \end{pmatrix}$$

Theorem 3.4 thus implies that $F(x)$ and $G(x)$ are finite sets for each x if this determinant is non-zero—a result consistent with our knowledge of finite dimensional linear systems theory. In the next section we apply our results to a more interesting (nonlinear) example.

4. A PROBLEM IN RIGID BODY CONTROL

The following system describes the rotation of a rigid body steered by a pair of opposing gas jets and whose rotation about one fixed axis can be observed.

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{pmatrix} = \begin{pmatrix} a_1 & x_2(t) & x_3(t) \\ a_2 & x_1(t) & x_3(t) \\ a_3 & x_1(t) & x_2(t) \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} u(t), \tag{5}$$

$$y(t) = c_1 x_1(t) + c_2 x_2(t) + c_3 x_3(t), \tag{6}$$

$$a_1 + a_2 + a_3 + a_1 a_2 a_3 = 0,$$

$$c_1^2 + c_2^2 + c_3^3 = 1,$$

$$-1 \leqslant a_i \leqslant 1.$$

If the principal moments of inertia are I_1, I_2 and I_3 then $a_1 = (I_2 - I_3)/I_1$, $a_2 = (I_3 - I_1)/I_2$, $a_3 = (I_1 - I_2)/I_3$. $x_i(t)$ is the angular velocity about the ith principal axis of inertia and $y(t)$ is the component of angular velocity about the axis (c_1, c_2, c_3) (written with respect to the principal axis coordinate system). $I_i b_i u(t)$ represents the torque applied about the ith axis by the gas jet. It is known that the set of initial conditions for (5) which give rise to periodic motions comprise a Zariski open subset of $\mathbb{R}^3$ (see, for example, [1]). It follows from an argument very similar to the one used for the proof of Theorem 4 in [2] that if (5) has the strong accessibility property on $\mathbb{R}^3$, then the set of points attainable in finite time (from any initial point) is all of $\mathbb{R}^3$.

To determine conditions under which this system has the strong accessibility property, according to theorem 2.2, we must calculate the linear span of $b, d^2f(b, b), d^2f(b, d^2f(b, b))$ and $d^2f(d^2f(b, b), d^2f(b, b))$ where

$$b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

and for any $v, w \in \mathbb{R}^3$,

$$d^2f\left(\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}\right) = \begin{pmatrix} a_1(v_2w_3 + v_3w_2) \\ a_2(v_1w_3 + v_3w_1) \\ a_3(v_1w_2 + v_2w_1 \end{pmatrix}.$$

The first three span $\mathbb{R}^3$ if and only if the determinant

$$\begin{vmatrix} b_1 & a_1b_2b_3 & a_1b_1(a_3b_2^2 + a_2b_3^2) \\ b_2 & a_2b_1b_3 & a_2b_2(a_3b_1^2 + a_1b_3^2) \\ b_3 & a_3b_1b_2 & a_3b_3(a_2b_1^2 + a_1b_2^2) \end{vmatrix} = (a_1b_2^2 - a_2b_1^2)(a_1b_3^2 - a_3b_1^2)(a_2b_3^2 - a_3b_2^2)$$

is non-zero. The fourth vector in the above list is

$$d^2f(d^2f(b, b), d^2f(b, b)) = \lambda \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

where $\lambda = 4a_1a_2a_3b_1b_2b_3$. It is now not difficult to see these four vectors span $\mathbb{R}^3$ precisely when we *don't* have $a_ib_j^2 = a_jb_i^2$ for some $i \neq j$. We summarize these observations as follows.

THEOREM 4.1. The system (5) has the strong accessibility property on $\mathbb{R}^3$ if and only if we *don't* have $a_ib_j^2 = a_jb_i^2$ for some $i \neq j$.

COROLLARY 4.1. For the system (5) the set of points attainable from x is $\mathbb{R}^3$ for all $x \in \mathbb{R}^3$ if and only if we *don't* have $a_ib_j^2 = a_jb_i^2$ for some $i \neq j$.

To assess the observability of this system we first compute $\mathscr{F}$.

$$h(s) = c_1s_1 + c_2s_2 + c_3s_3, \quad L_fh(s) = c_1a_1s_2s_3 + c_2a_2s_1s_3 + c_3a_3s_1s_2$$

and

$$L_f^2h(s) = c_1a_1a_3s_1s_2^2 + c_2a_2a_3s_1^2s_2 + c_1a_1a_2s_1s_3^2 + c_3a_2a_3s_1^2s_3 + c_2a_1a_2s_2s_3^2 + c_3a_1a_3s_2^2s_3.$$

A straightforward calculation shows that these three polynomials form a basis for the ideal $\mathscr{V}_{\mathscr{F}}$. To employ Theorem 3.4 we therefore compute the inertia forms of the system of equations $h(x) = 0$, $L_fh(x) = 0$ and $L_f^2h(x) = 0$. One inertia form suffices for this system, and a somewhat tedious calculation using elimination theory shows it may be taken to be

$$\begin{aligned}\phi(a_1, a_2, a_3, c_1, c_2, c_3) = {} & [(a_1c_1^2 - a_2c_2^2)^4 + (a_1c_1^2 - a_3c_3^2)^4 \\ & + (a_2c_2^2 - a_3c_3^2)^4 - a_1^4c_1^8 - a_2^4c_2^8 - a_3^4c_3^8 \\ & + 4a_1a_2a_3c_1^2c_2^2c_3^2(a_1c_1^2 + a_2c_2^2 + a_3c_3^2)]a_1a_2a^3c_1c_2c_3.\end{aligned}$$

THEOREM 4.2. Suppose the output (6) of the free motion ($u(t) \equiv 0$) of (5) is observed on some interval $[0, \delta]$. The condition $a_1a_2a_3c_1c_2c_3 \neq 0$ is necessary and sufficient for all initial states to be distinguishable from the zero initial state and also for all but finitely many initial states to be distinguishable from any initial state $x_0 \in \mathbb{R}^3$.

Proof. In light of Corollary 3.1 we need to show $\phi(a_1, a_2, a_3, c_1, c_2, c_3) \neq 0$ if and only if $a_1 a_2 a_3 c_1 c_2 c_3 \neq 0$. If $a_1 a_2 a_3 c_1 c_2 c_3 \neq 0$ then $\phi(a_1, a_2, a_3, c_1, c_2, c_3)$ can be zero only if

$$(a_1 c_1^2 - a_2 c_2^2)^4 + (a_1 c_1^2 - a_3 c_3^2)^4 + (a_2 c_2^2 - a_3 c_3^2)^4 \\ - a_1^4 c_1^8 - a_2^4 c_2^8 - a_3^4 c_3^8 + 4 a_1 a_2 a_3 c_1^2 c_2^2 c_3^2 = 0.$$

Since $a_1 + a_2 + a_3 + a_1 a_2 a_3 = 0$, two of the a_i's must be of one sign while the remaining a_i has the opposite sign. Let $\alpha_1 = a_1 c_1^2, \alpha_2 = a_2 c_2^2, \alpha_3 = a_3 c_3^2$ and assume, with no loss of generality, that α_1 and α_2 have the same sign with α_3 having the opposite sign. Rewrite the polynomial as

$$(\alpha_1 - \alpha_2)^4 + (\alpha_1 - \alpha_3)^4 + (\alpha_2 - \alpha_3)^4 - \alpha_1^4 - \alpha_2^4 - \alpha_3^4 + 4\alpha_1\alpha_2\alpha_3(\alpha_1 + \alpha_2 + \alpha_3)$$

which may be rewritten as

$$(\alpha_1 - \alpha_2)^4 + \alpha_3^4 + 6(\alpha_1^2 + \alpha_2^2)\alpha_3^2 - 4(\alpha_1 - \alpha_2)^2(\alpha_1 - \alpha_2)\alpha_3 + 4(\alpha_1\alpha_2 - (\alpha_1 + \alpha_2)\alpha_3)\alpha_3^2.$$

Since α_1 and α_2 have the same sign and α_3 has the opposite sign the fourth term is nonnegative, while the last term is positive. The first three terms are nonnegative, so that the whole expression is positive. This implies that $\phi(a_1, a_2, a_3, c_1, c_2, c_3) \neq 0$ and the conclusion of the theorem follows from Theorem 3.4.

A similar analysis for the case of nonzero inputs is clearly possible, but we omit the details.

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