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Control of Ensemble Systems on Special Orthogonal Groups

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***Abstract*— In this paper, we study the control of an ensemble of time-invariant bilinear systems defined on the special orthogonal group SO(n). This type of ensemble control systems appears in various application domains, such as the manipulation of quantum spin ensembles and motion planning for a population of robots. We establish an explicit algebraic necessary and sufficient condition to examine the controllability of systems on SO(n) by using the terminology from the theory of symmetric groups, which provides a transparent means to analyze the underlying Lie algebras. In addition, we show the equivalence between controllability and ensemble controllability of individual and ensemble systems, respectively, for systems evolving on SO(n).**

I. INTRODUCTION

Problems involving the control of an ensemble of structurally identical systems defined on Lie groups have been the subject of numerous recent investigations. Primary interests arise from multidisciplinary domains, such as exciting an ensemble of quantum systems for applications of spectroscopy, imaging, and computation [1, 2, 3] and coordinating the movement of flocks or a population of robots in biology and robotics [4, 5]. Although these problems are well motivated and highly deserved to study, there existed little progress in theoretical developments towards a complete understanding of fundamental properties of ensemble control systems on Lie groups, and most state-of-the-art results were centered around the ensemble systems defined on SO(3) [6, 7, 8].

In this paper, we characterize the controllability of ensemble systems defined on the general special orthogonal group SO(n). In particular, we develop an effective algebraic approach for examining controllability by mapping the Lie algebra operations to the permutation operations in symmetric groups. Moreover, we analyze controllability through the procedure of covering the Lie algebra $\mathfrak{so}(n)$ by its sub-algebras that are isomorphic to $\mathfrak{so}(3)$. In this way, the ensemble controllability analysis can be systematically reduced to that of SO(3), which was established in our previous work [6].

In the following section, we will review the notion of ensemble controllability and summarize the previous work on ensemble control of systems on the Lie group SO(3). We will then extend these previous results to the case in which the ensemble system on SO(3) consists of a general structure of parameter variations. This facilitates our controllability analysis for ensemble systems on SO(n). In Section III, we use the terminology of symmetric groups to construct an algebraic necessary and sufficient controllability condition for a single as well as an ensemble of systems on SO(n). We then reveal the equivalence of controllability and ensemble controllability, respectively, for individual and ensemble systems on SO(n). Examples are provided to illustrate the advantage and effectiveness of our approach.

II. ENSEMBLE CONTROL ON SO(3)

In this section, we define the notion of ensemble controllability and review the previous results on controllability of an ensemble systems defined on the special orthogonal group SO(3) [6], which lay the foundation for analyzing controllability for the ensemble systems on SO(n).

A. Control of Ensemble Systems

Consider a parameterized family of control systems

$$\frac{d}{dt}x(t,\varepsilon) = f(t, x(t,\varepsilon), u(t)) \tag{1}$$

defined on a state space M and indexed by the parameter ε taking values on some compact set $K \subset \mathbb{R}^d$. The same control $u(t) \in \mathbb{R}^m$ is being used to simultaneously steer this family of control systems. For such systems, we define the notion of ensemble controllability as follows.

Definition 1. *Let $\mathcal{F}(K)$ denote the space of M-valued functions defined on K. The family of systems in* (1) *is said to be ensemble controllable on the function space $\mathcal{F}(K)$, if for any $\delta > 0$ and starting with any initial state $x_0 \in \mathcal{F}(K)$, where $x_0(\varepsilon) = x(0,\varepsilon)$, there exists a control law $u(t)$ that steers the system into a δ-neighborhood of a desired target state $x_F \in \mathcal{F}(K)$ at a finite time $T > 0$, i.e., $\|x(T,\varepsilon) - x_F(\varepsilon)\| < \delta$, where $\|\cdot\|$ is a norm on $\mathcal{F}(K)$. Note the final time T may depend on δ.*

In this work, specifically, we study time-invariant driftless ensemble systems defined on a Lie group G of the form

$$\frac{d}{dt}X(t,\varepsilon) = \left[\sum_{i=1}^{m} \varepsilon_i u_i(t) B_i\right] X(t,\varepsilon), X(0,\varepsilon) = I \tag{2}$$

where $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_m)' \in K$, a compact subset of $\mathbb{R}^m$, the state $X(t,\cdot) \in C(K,G)$, the space of continuous G-valued functions under the supreme norm, for each $t \geq 0$, and $B_1, \ldots, B_m$ are linearly independent elements in the Lie algebra $\mathfrak{g}$ of G, I is the identity matrix, and $u_i(t) \in \mathbb{R}, i = 1, \ldots, m$ are piecewise continuous control functions.

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B. Controllability of Ensemble Systems on SO(3)

Manipulating an ensemble system evolving on SO(3) is an important problem in the area of quantum control, which models practical control design problems in nuclear magnetic resonance (NMR) spectroscopy and imaging (MRI), quantum computation, and quantum information processing [1, 3]. In this case, we consider $G = \mathrm{SO}(3)$ with $K = [a, b] \subset \mathbb{R}^+$, and the ensemble system is expressed in the form as in (2), given by

$$\frac{d}{dt}X(t,\varepsilon) = \varepsilon\big[u\Omega_y + v\Omega_x\big]X(t,\varepsilon), X(0,\varepsilon) = I \quad (3)$$

where

$$\Omega_y = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad \Omega_x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$

We know that $C(K, \mathrm{SO}(3))$ itself is an infinite-dimensional Lie group with the Lie algebra $C(K, \mathfrak{so}(3))$, where $\mathfrak{so}(3)$ is the Lie algebra of SO(3). Let $f_n(\varepsilon) = (\varepsilon/b)^n$, then the sequence $f_n(\varepsilon)\Omega_x$ in $C(K, \mathfrak{so}(3))$ has no convergent subsequence, and thus $C(K, \mathrm{SO}(3))$ is not compact, which makes the study of ensemble controllability challenging. Because $\mathfrak{so}(3)$ is a 3-dimensional vector space over $\mathbb{R}$, $C(K, \mathfrak{so}(3))$ can be seen as a 3-dimensional vector space over $C(K, \mathbb{R})$. Adopting this idea, we can analyze ensemble controllability of the system (3).

Lemma 1. *The system* (3) *is ensemble controllable on* $C(K, \mathrm{SO}(3))$ *[6].*

Proof. Observe that the Lie brackets generated by the set of matrices $\{\varepsilon\Omega_y, \varepsilon\Omega_x\}$ are

$$\mathrm{ad}^{2k+1}_{\varepsilon\Omega_y}(\varepsilon\Omega_x) = (-1)^k\varepsilon^{2k}\Omega_z,$$
$$\mathrm{ad}^{2k}_{\varepsilon\Omega_y}(\varepsilon\Omega_x) = (-1)^k\varepsilon^{2k+1}\Omega_x,$$

where $k \in \mathbb{N}$ and $\mathrm{ad}_A B = [A, B]$ for all $A, B \in \mathfrak{so}(3)$. Now using $\{\varepsilon\Omega_x, \varepsilon^3\Omega_x, \ldots, \varepsilon^{2n+1}\Omega_x\}$ as generators, we are able to produce an evolution of the form

$$R_x(\varepsilon) = \exp(c_0\varepsilon\Omega_x)\exp(c_1\varepsilon^3\Omega_x)\cdots\exp(c_n\varepsilon^{2n+1}\Omega_x) = \exp\left\{\sum_{k=0}^{n} c_k\varepsilon^{2k+1}\Omega_x\right\}. \quad (4)$$

As a result, given any ε-dependent rotation around x-axis, i.e., $\exp\{\theta(\varepsilon)\Omega_x\}$ where $\theta(\varepsilon) \in C(K)$, the order of the polynomial n and the coefficients c_k can be appropriately chosen so that $\sum_{k=0}^{n} c_k\varepsilon^{2k+1} \approx \theta_x(\varepsilon)$ for all $\varepsilon \in K$, i.e, $\|\sum_{k=0}^{n} c_k\varepsilon^{2k+1} - \theta_x(\varepsilon)\|_\infty < \delta$. Similar arguments can be developed to show that we can approximately generate any ε-dependent rotation around y-axis, i.e., $\exp\{\theta_y(\varepsilon)\Omega_y\}$, and hence produce any three-dimensional rotation. Namely, given any rotation $\Theta(\varepsilon) \in C(K, \mathfrak{so}(3))$, we can parameterize it by the Euler angles (α, β, γ) such that

$$\Theta(\varepsilon) = \exp\{\alpha(\varepsilon)\Omega_x\}\exp\{\beta(\varepsilon)\Omega_y\}\exp\{\gamma(\varepsilon)\Omega_x\},$$

and then the desired rotations characterized by the continuous functions $\alpha, \beta, \gamma \in C(K)$ can be synthesized by using the control vector fields as described in (4). This concludes that the system in (3) is ensemble controllable (or approximately controllable) with respect to the topology of uniform convergence. □

It was also shown in our previous work that the ensemble with a dispersion in the drift, i.e., the system

$$\frac{d}{dt}X(t,\varepsilon,\omega) = \big[\omega\Omega_z + \varepsilon u\Omega_y + \varepsilon v\Omega_z\big]X(t,\varepsilon,\omega), \quad (5)$$

where $\omega \in K' \subset \mathbb{R}$ with K' compact and

$$\Omega_z = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

is ensemble controllable on $C(K \times K', \mathrm{SO}(3))$ [6]. Next, we extend these previous results to investigate the ensemble system on SO(3) with three parameter variations.

Proposition 1. *Let K be a compact subset of $(\mathbb{R}^+)^3$, then the ensemble systems*

$$\frac{d}{dt}X(t,\varepsilon) = \big[\varepsilon_1 u_1\Omega_x + \varepsilon_2 u_2\Omega_y + \varepsilon_3 u_3\Omega_z\big]X(t,\varepsilon) \quad (6)$$

is ensemble controllable on $C(K, \mathrm{SO}(3))$, where $\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3)' \in K$ and $u_i : [0, T] \to \mathbb{R}$, $i = 1, 2, 3$, are piecewise continuous.

Proof. By successive Lie bracketing of the set of elements $\{\varepsilon_2\Omega_y, \varepsilon_3\Omega_z\}$, we obtain

$$\mathrm{ad}^{2k+1}_{\varepsilon_2\Omega_y}(\varepsilon_3\Omega_z) = (-1)^k\varepsilon_2^{2k+1}\varepsilon_3\Omega_x,$$
$$\mathrm{ad}^{2l+1}_{\varepsilon_3\Omega_z}(\varepsilon_2^{2k+1}\varepsilon_3\Omega_x) = (-1)^l\varepsilon_2^{2k+1}\varepsilon_3^{2l+1}\Omega_x,$$

where $k, l = 0, 1, 2, \ldots$. Defining $L_{(k,l)} = \varepsilon_2^{2k+1}\varepsilon_3^{2l+1}$ and applying the iterated Lie brackets for $L_{(k,l)}\Omega_x$ yields

$$\mathrm{ad}^{2s}_{[\varepsilon_1\Omega_x,\varepsilon_2\Omega_y]}(L_{(k,l)}\Omega_x) = (-1)^s\varepsilon_1^{2s}\varepsilon_2^{2(k+s)+1}\varepsilon_3^{2l+1}\Omega_x = (-1)^s\varepsilon_1^{2s}\varepsilon_2^{2(k+s)}\varepsilon_3^{2l}(\varepsilon_2\varepsilon_3\Omega_x),$$

where $s = 0, 1, 2, \ldots$. Furthermore, let $L_{(s,k,l)}(\varepsilon) = \varepsilon_1^{2s}\varepsilon_2^{2(k+s)}\varepsilon_3^{2l}$ and $\mathcal{A} = \mathrm{span}\{L_{(s,k,l)} : s, k, l = 0, 1, \ldots\} \subset C(K, \mathbb{R})$, then for any functions $f, g \in \mathcal{A}$, their product fg also belongs to $\mathcal{A}$, so $\mathcal{A}$ is a subalgebra of $C(K, \mathbb{R})$. Pick any two points $x = (x_1, x_2, x_3)'$ and $y = (y_1, y_2, y_3)'$ in K and assume that $f(x) = f(y)$ for all $f \in \mathcal{A}$, in particular, taking $L_{(1,0,0)}(x) = L_{(1,0,0)}(y)$, $L_{(0,1,0)}(x) = L_{(0,1,0)}(y)$ and $L_{(0,0,1)}(x) = L_{(0,0,1)}(y)$, then we obtain $x_i = y_i$ for each $i = 1, 2, 3$, namely, $x = y$. Therefore, $\mathcal{A}$ separates points in K, which implies that $\mathcal{A}$ is dense in $C(K, \mathbb{R})$ by the Stone-Weierstrass Theorem (see Theorem 4 in Appendix). In other words, for any $f \in C(K, \mathbb{R})$, we can uniformly approximate $f(\varepsilon)\Omega_x$ by the iterated Lie brackets of the elements in $\{\varepsilon_1\Omega_x, \varepsilon_2\Omega_y, \varepsilon_3\Omega_z\}$. Similarly, we can prove that for any $g, h \in C(K, \mathbb{R})$, $g(\varepsilon)\Omega_y$ and $h(\varepsilon)\Omega_z$ can also be uniformly approximated by the same procedure. It follows that $\overline{\mathrm{Lie}\{\varepsilon_1\Omega_x, \varepsilon_2\Omega_y, \varepsilon_3\Omega_z\}} = C(K, \mathfrak{so}(3))$, where $\mathrm{Lie}\{\varepsilon_1\Omega_x, \varepsilon_2\Omega_y, \varepsilon_3\Omega_z\}$ denote the Lie algebra generate by the set $\{\varepsilon_1\Omega_x, \varepsilon_2\Omega_y, \varepsilon_3\Omega_z\}$ and $\overline{\mathrm{Lie}\{\varepsilon_1\Omega_x, \varepsilon_2\Omega_y, \varepsilon_3\Omega_z\}}$ denote its closure under the topology of uniform convergence, which implies that the system in (6) is ensemble controllable on $C(K, \mathrm{SO}(3))$. $\square$

It is worth mentioning that the result of Theorem 1 still holds if K is a locally compact subset of $(\mathbb{R}^+)^3$, because the condition of the Stone-Weierstrass Theorem can be relaxed to functions defined on locally compact domains.

In the next section, we will utilize these results on SO(3) and carry out an extension to analyze the controllability of ensemble systems on SO(n) for $n \geq 3$.

III. Ensemble Control on SO(n)

A. A Review of the Lie Algebra $\mathfrak{so}(n)$

Let $E_{ij} \in \mathbb{R}^{n\times n}$ denote the matrix whose ij^{th} entry is 1 and others are 0, and define $\Omega_{ij} = E_{ij} - E_{ji}$, then Ω_{ij} satisfies $\Omega_{ij} = -\Omega_{ji}$ if $i \neq j$ and $\Omega_{ij} = 0$ if $i = j$. Furthermore, the set $\mathcal{B} = \{\Omega_{ij} : 1 \leq i < j \leq n\}$ forms a basis of $\mathfrak{so}(n)$, and we call $\mathcal{B}$ the standard basis of $\mathfrak{so}(n)$. The following lemma characterizes the Lie bracket relations on the standard basis elements of $\mathfrak{so}(n)$.

Lemma 2. *The Lie brackets between the elements in $\mathcal{B}$ satisfy the relations $[\Omega_{ij}, \Omega_{kl}] = \delta_{jk}\Omega_{il} + \delta_{il}\Omega_{jk} + \delta_{jl}\Omega_{ki} + \delta_{ik}\Omega_{lj}$, where δ is the Kronecker delta function, i.e.,*

$$\delta_{mn} = \begin{cases} 1 & \text{if } m = n, \\ 0 & \text{if } m \neq n. \end{cases}$$

Proof. Notice that $E_{ij}E_{kl} = \delta_{jk}E_{il}$, so $[E_{ij}, E_{kl}] = \delta_{jk}E_{il} - \delta_{li}E_{kj}$. Following the bilinearity of the Lie bracket, we get

$$\begin{aligned}
[\Omega_{ij}, \Omega_{kl}] &= [E_{ij} - E_{ji}, E_{kl} - E_{lk}] \\
&= [E_{ij}, E_{kl}] - [E_{ij}, E_{lk}] - [E_{ji}, E_{kl}] + [E_{ji}, E_{lk}] \\
&= \delta_{jk}E_{il} - \delta_{li}E_{kj} - \delta_{jl}E_{ik} + \delta_{ki}E_{lj} \\
&\quad - \delta_{ik}E_{jl} + \delta_{lj}E_{ki} + \delta_{il}E_{jk} - \delta_{kj}E_{li} \\
&= \delta_{jk}\Omega_{il} + \delta_{il}\Omega_{jk} + \delta_{jl}\Omega_{ki} + \delta_{ik}\Omega_{lj}.
\end{aligned}$$

$\square$

According to Lemma 2, for any $\Omega_{ij}, \Omega_{kl} \in \mathcal{B}$, $[\Omega_{ij}, \Omega_{kl}] \neq 0$ if and only if exactly one of the following equalities, $i = l$ $j = k$, $i = k$, $j = l$, holds.

B. Controllability of Systems on SO(n)

Controllability of systems evolving on compact, connected Lie groups has been extensively studied [9, 10, 11, 12]. The central idea is with regard to whether the Lie algebra generated by the control (and the drift) vector fields is equivalent to the underlying Lie algebra of the Lie group. Here, we recap the controllability results for a driftless system defined on SO(n), given by

$$\frac{d}{dt}X(t) = \left[\sum_{i=1}^{m'} u_i(t)B_i\right] X(t), X(0) = I, \qquad (7)$$

where $X(t) \in \mathrm{SO}(n)$ is the state, $B_i \in \mathfrak{so}(n)$ are linearly independent, and $u_i(t) \in \mathbb{R}$ are the control functions. Because $\mathrm{Lie}(\{B_i : i = 1, \ldots, m\})$ is a Lie subalgebra of $\mathfrak{so}(n)$, it is a Lie algebra whose dimension is less than or equal to the dimension of $\mathfrak{so}(n)$, that is, $n(n-1)/2$. Then, if the dimension of $\mathrm{Lie}(\{B_i : i = 1, \ldots, m\})$ is $n(n-1)/2$, then $\mathrm{Lie}(\{B_i : i = 1, \ldots, m\}) = \mathfrak{so}(n)$ holds, which implies the system in (7) is controllable on SO(n) by the Lie algebra rank condition (see Theorem 3 in Appendix). Specifically, we will forcus on systems on SO(n) in the form of

$$\frac{d}{dt}X(t) = \left[\sum_{k=1}^{m} u_k(t)\Omega_{i_k,j_k}\right] X(t), X(0) = I, \qquad (8)$$

where $\Omega_{i_k,j_k} \in \mathcal{B}$ for all $k = 1, \ldots, m$.

In order to facilitate our development of controllability conditions for the system in (8), we use the elements of the symmetric group on n symbols S_n to represent the subsets of $\mathcal{B}$, i.e., the standard basis of $\mathfrak{so}(n)$. Recall that every element $\sigma \in S_n$ is a permutation on n letters, i.e., a bijective map $\sigma : Z_n \to Z_n$, where Z_n is a set containing n elements, conventionally, $Z_n = \{1, \ldots, n\}$. In addition, an equivalence relation on Z_n can be defined by $a \sim b$ if and only if $b = \sigma^k(a)$ for some $k \in \mathbb{Z}$, where $a, b \in Z_n$. The equivalence classes in Z_n determined by this equivalence relation is called the orbits of σ. A permutation $\sigma \in S_n$ is a cycle if it has at most one orbit containing more than one element, and the length of a cycle is the number of elements in its largest orbit. A cycle of length 2 is called a transposition. Any permutation of a finite set of at least two elements is a product of transpositions [13].

Now, we will identify each subset of $\mathcal{B}$ with an element in S_n. Let $\mathcal{P}(\mathcal{B})$ denote the power set of $\mathcal{B}$, and define a map $\iota : \mathcal{P}(\mathcal{B}) \to S_n$ by $\{\Omega_{i_1,j_1}, \ldots, \Omega_{i_k,j_k}\} \mapsto (i_k, j_k) \cdots (i_1, j_1)$, where (i_s, j_s) is the cyclic notation of the permutation

$$\begin{pmatrix} 1 & \cdots & i_s & \ldots & j_s & \ldots & n \\ 1 & \cdots & j_s & \ldots & i_s & \ldots & n \end{pmatrix}$$

for each $s = 1, \ldots, k$.

Lemma 3. *The map $\iota : \mathcal{P}(\mathcal{B}) \to S_n$ is surjective.*

Proof. Because any permutation $\sigma \in S_n$ can be written as a product of transpositions [13], there exist $1 \leq i_1, j_1, \ldots, i_m, j_m \leq n$ such that $i_k < j_k$ for all $k = 1, \ldots, m$ and $\sigma = (i_m, j_m) \ldots (i_1, j_1)$. Let $\mathcal{S} = \{\Omega_{i_1,j_1}, \ldots, \Omega_{i_m,j_m}\}$, we then have $\iota(\mathcal{S}) = \sigma$ by the definition of the map ι. Since $\sigma \in S_n$ is arbitrary, the map ι is surjective. $\square$

Using the terminologies from the symmetric group theory introduced above, we can algebraically construct a necessary and sufficient controllability condition for the systems defined on SO(n) in the form of (8). This will provide powerful machinery for analyzing controllability of the ensemble systems on SO(n), which will be described in Section III-C.

Theorem 1. *The control system on* SO(n) *as in* (8) *is controllable if and only if there is a subset $\mathcal{S}$ of $\{\Omega_{i_1,j_1}, \ldots, \Omega_{i_m,j_m}\}$ such that $\iota(\mathcal{S})$ is a cycle of length n.*

Before proving the theorem, we will explore the relationship between the Lie bracket operation on $\mathcal{B}$ and the product operation on S_n. For any $\Omega_{ij}, \Omega_{kl} \in \mathcal{B}$, by Lemma 2, we have $[\Omega_{ij}, \Omega_{kl}] \neq 0$ if and only if exactly one of the following equalities, $i = l$ $j = k$, $i = k$, $j = l$, holds, say $j = k$, then $[\Omega_{ij}, \Omega_{kl}] = \Omega_{il}$. In this situation, $\iota(\Omega_{kl})\iota(\Omega_{ij}) = \iota(\Omega_{jl})\iota(\Omega_{ij}) = (j, l)(i, j) = (i, j, l)$. On the other hand, if $[\Omega_{ij}, \Omega_{kl}] = 0$, then there are two cases: (i) $i = k$ and $j = l$, then $(i, j)(k, l) = e$, where $e \in S_n$ denotes the identity map; (ii) i, j, k, l are all distinct, then $(i, j)(k, l)$ is a permutation as a product of two disjoint transpositions.

Remark 1. According to the relationship between the Lie bracket operation on $\mathcal{B}$ and the product operation on S_n described above, we reach the following conclusions:

1) Under the map ι, Lie bracketing elements in $\mathcal{B}$ iteratively will increase the length of the cycle if the result of the iterated Lie brackets is nontrivial and each iteration generates a new element. More precisely, in this situation, l times iterated Lie bracket operations will increase the length of the cycle by l.
2) If the image of a subset $\mathcal{S}$ of $\mathcal{B}$, where $\mathcal{S} = \{\Omega_{i_1,j_1}, \ldots, \Omega_{i_m,j_m}\}$, contains a cycle with length greater than or equal to J, then the cardinality of the index set $J = \{i_1, j_1, \ldots, i_m, j_m\}$, which is denoted by $|I|$, must satisfy $l \leq |I| \leq 2m - (l - 2)$.

Example 1. Let $\mathcal{B}_4 = \{\Omega_{ij} : 1 \leq i < j \leq 4\}$ be the set of standard basis for the Lie algebra $\mathfrak{so}(4)$. Consider the subset $\mathcal{S} = \{\Omega_{12}, \Omega_{23}, \Omega_{34}\}$ of $\mathcal{B}_4$. Because $\mathcal{S}$ contains 3 elements, we can only proceed iterated Lie brackets with different elements in $\mathcal{S}$ twice. By Lemma 1, we have $[\Omega_{12}, \Omega_{23}] = \Omega_{13}$ and $[[\Omega_{12}, \Omega_{23}], \Omega_{34}] = \Omega_{14}$, which also shows that each iteration of the successive Lie brackets results in a nonzero and non-repeating element of $\mathcal{B}_4$. In addition, under the map ι, each element in $\mathcal{S}$ corresponds to a transposition, i.e., a cycle of length 2, and taking successive Lie bracketing for two times increases the length of a transposition by 2. Hence, the image of $\mathcal{S}$ under ι should be a cycle of length 4. We know that $\iota(\mathcal{S}) = (3, 4)(2, 3)(1, 2) = (1, 2, 3, 4)$ which is exactly a cycle of length 4, thus the first conclusion of Remark 1 is verified. If we proceed Lie bracketing with the element in $\mathcal{S}$ one more time, e.g., $[\Omega_{14}, \Omega_{13}] = \Omega_{34}$, which is already in $\mathcal{S}$, then under the map ι, we have $(3, 4)(1, 2, 3, 4) = (1, 2, 4)$, and the length of the cycle is decreased. Hence, in order to increase the length of the cycle, the condition of generating a new element at each iteration of Lie bracketing is essential.

Now, we extend $\mathcal{S}$ to $\mathcal{S}' = \mathcal{S} \cup \{\Omega_{24}\}$. Because $\Omega_{24} = [\Omega_{23}, \Omega_{34}]$ and $\Omega_{23}, \Omega_{34} \in \mathcal{S}$, Ω_{24} can be generated by iterated Lie brackets of the elements in the proper subset $\mathcal{S}$ of $\mathcal{S}'$, and thus Lie($\mathcal{S}$)=Lie($\mathcal{S}'$). In other words, Ω_{24} is redundant when we study the Lie algebra generated by $\mathcal{S}'$. Applying the definition of the map ι to $\mathcal{S}'$, we obtain $\iota(\mathcal{S}') = \iota(\Omega_{24})\iota(\mathcal{S}) = (2, 4)(1, 2, 3, 4) = (1, 4)(2, 3)$ which is no longer a cycle. Therefore, Lie bracketing redundant elements does not result in a increase of the length of the cycle either. In addition, every cycle in S_4 must have length less than or equal to 4, and thus a subset of $\mathcal{B}_4$ with 3 elements is enough to study the Lie algebra $\mathfrak{so}(4)$.

Next, if we remove one element from $\mathcal{S}$, e.g., define $\mathcal{S}'' = \mathcal{S} \backslash \{\Omega_{34}\}$, then the index set associated with $\mathcal{S}''$ is $J_{\mathcal{S}''} = \{1, 2, 2, 3\} = \{1, 2, 3\}$ whose cardinality is $|J_{\mathcal{S}''}| = 3$. According to the second conclusion of Remark 1, $\iota(\mathcal{S}'')$ cannot be a cycle of length greater than 3. By the definition of ι, $\iota(\mathcal{S}'') = (1, 2, 3)$ which is exactly a cycle of length 3. If we replace Ω_{23} by Ω_{34} in $\mathcal{S}''$, namely, $\mathcal{S}'' = \{\Omega_{12}, \Omega_{34}\}$ now, then $|J_{\mathcal{S}''}| = 4$. However, in this case, we cannot produce a cycle of length 3 either, because $2m - (l - 2) = 2 \times 2 - (3 - 2) = 3 < 4$, which violates 2) in Remark 1. As a result, $\iota(\mathcal{S}'') = (1, 2)(3, 4)$ and $[\Omega_{12}, \Omega_{34}] = 0$.

Next, we will prove Theorem 1.

(Proof of Theorem 1): The system in (8) is controllable on SO(n) if and only if Lie$(\{\Omega_{i_1,j_2}, \ldots, \Omega_{i_m,j_m}\}) = \mathfrak{so}(n)$ (see Theorem 3 in Appendix), therefore, it is equivalent to show that Lie$(\mathcal{S}) = \mathfrak{so}(n)$ if and only if $\iota(\mathcal{S})$ is a cycle of length n for some subsets $\mathcal{S}$ of $\{\Omega_{i_1,j_2}, \ldots, \Omega_{i_m,j_m}\}$. Because a cycle of length n can be decomposed into a product of at least $n-1$ transpositions, m has to be greater than or equal to $n-1$. So, it suffices to assume that the cardinality of $\mathcal{S}$ is $n - 1$, and, without loss of generality, let $\mathcal{S} = \{\Omega_{i_1,j_1}, \ldots, \Omega_{i_{n-1},j_{n-1}}\}$.

(Necessity): if the system in (8) is controllable, then Lie$(\mathcal{S}) = \mathfrak{so}(n)$, and thus any element in $\mathcal{B}$ can be generated by the iterated Lie brackets of elements in $\mathcal{S}$. Pick $\Omega_{st} \in \mathcal{B}\backslash\mathcal{S}$, there exist $\Omega_{i_1,j_1}, \ldots, \Omega_{i_l,j_l}, l \leq n - 1$ such that $\prod_{k=l}^{2} \mathrm{ad}_{\Omega_{i_k,j_k}}(\Omega_{i_1,j_1}) = \pm\Omega_{st}$, and so $[\Omega_{i_k,j_k}, \Omega_{i_{k+1},j_{k+1}}] \neq 0$ for all $k = 1, \ldots, l - 1$. By Remark 1, $\sigma = \prod_{k=l}^{1} \iota(\Omega_{i_k,j_k})$ is a cycle of length $l+1$. If $l = n-1$, then we are done. If $l < n - 1$, then the index set $J = \{i_1, j_1, \ldots, i_l, j_l\}$ is a proper subset of $Z_n = \{1, \ldots, n\}$. Pick distinct elements of $\mathcal{B}$, say $\Omega_{s_1,t_1}, \ldots, \Omega_{s_{n-l-1},t_{n-l-1}}$, such that $s_1 = t$ and $t_{k+1} = s_k \in Z_n\backslash J$ for all $k = 2, \ldots, n-l-2$. In addition, we also request $t_k \neq t_{k+1}$ for all $k = 1, \ldots, n-l-2$. Then, there are $\Omega_{i_{l+1},j_{l+1}}, \ldots, \Omega_{i_n,j_n} \in \mathcal{S}\backslash\{\Omega_{i_1,j_1}, \ldots \Omega_{i_l,j_l}\}$ such that

$$\prod_{k=l+1+a}^{l+1} \mathrm{ad}_{\Omega_{i_k,j_k}}(\Omega_{ij}) = \prod_{k=a}^{1} \mathrm{ad}_{\Omega_{s_i,t_i}}(\Omega_{ij})$$

holds for all $a = 1, \ldots, n-l-1$. By Remark 1, we have $\sigma' = \prod_{k=l+1+a}^{l+1}(\Omega_{i_k,j_k})$ that is a cycle of length $n-l$. Notice that $\{j\} = J \cap J'$, where $J' = \{i_{l+1}, j_{l+1}, \ldots, i_n, j_n\}$, and hence σ and σ' are not disjoint. Therefore, $\sigma'\sigma = \prod_{k=1}^{n} \iota(\Omega_{i_k,j_k})$ is a cycle of length n.

(Sufficiency): Assume that $\iota(\mathcal{S})$ is a cycle of length n. According to Remark 1, the cardinality of the index set $J = \{i_1, j_1, \ldots, i_{n-1}, j_{n-1}\}$ satisfies $n \leq |J| \leq 2(n - 1) - (n - 2) = n$, so $|J| = n$, which implies $J = \{1, \ldots, n\}$. Then, for any $a, b \in J$ such that $a < b$ and $\Omega_{ab} \notin \mathcal{S}$, there are some indices in J equal to them. Without loss of generality, assume $i_s = a$ and $j_t = b$. If $\{i_s, j_s\} \cap \{i_t, j_t\} \neq \varnothing$, then $j_s = i_t$, and thus $[\Omega_{i_s,j_s}, \Omega_{i_t,j_t}] = \Omega_{i_s,j_t} = \Omega_{ab}$; otherwise, because $\iota(\mathcal{S})$ is a cycle of length n, there exists some $\Omega_{i_{k_1},j_{k_1}} \in \mathcal{S}$ such that $\{i_s, j_s\} \neq \{i_{k_1}, j_{k_1}\}$ and $\{i_s, j_s\} \cap \{i_{k_1}, j_{k_1}\} \neq \varnothing$, i.e., there is exactly one of the following equalities $i_s = i_{k_1}, i_s = j_{k_1}, j_s = i_{k_1}, j_s = j_{k_1}$ holds. Suppose such $\Omega_{i_{k_1},j_{k_1}}$ does not exist, then we have two cases: (1) $\{i_s, j_s\} = \{i_k, j_k\}$ for all $k = 1, \ldots, n-1$, then the index set $J = \{i_s, j_s\}$ has cardinality 2, which contradicts with the fact $|J| = n \geq 3$; and (2) $\{i_s, j_s\} \cap \{i_k, j_k\} = \varnothing$ for all $k = 1, \ldots, n-1$, so that $\iota(\Omega_{i_s,t_s}) = (i_s, j_s)$ is a transposition disjoint from $\iota(\mathcal{S}\backslash\{\Omega_{i_s,j_s}\})$, then $\iota(\mathcal{S})$ has at least two orbits and one of them is (i_s, j_s), which contradicts the assumption that $\iota(\mathcal{S})$ is a cycle with only one orbit.

Next, if $\{i_{k_1}, j_{k_1}\} \cap \{i_t, j_t\} \neq \varnothing$, then $[[\Omega_{i_s,j_s}, \Omega_{i_{k_1},j_{k_1}}], \Omega_{i_t,j_t}] = \pm\Omega_{ab}$. If $\{i_s, j_s\} \cap \{i_{k_1}, j_{k_1}\} = \varnothing$, then pick $\Omega_{i_{k_2},j_{k_2}} \in \mathcal{S}$ such that $\{i_{k_1}, j_{k_1}\} \neq \{i_{k_2}, j_{k_2}\}$ and $\{i_{k_1}, j_{k_1}\} \cap \{i_{k_2}, j_{k_2}\} \neq \varnothing$. Repeat this procedure until we get $\Omega_{i_{k_N},j_{k_N}} \in \mathcal{S}$ such that $\{i_{k_N}, j_{k_N}\} \cap \{i_t, j_t\} \neq \varnothing$. Because $\mathcal{S}$ is a finite set, this procedure will stop in finite steps. Let $s = k_0$ and $t = k_{N+1}$, and notice that the sequence of indices $\{i_{k_0}, j_{k_0}\}, \ldots, \{i_{k_{N+1}}, j_{k_{N+1}}\}$ are distinct and there is exactly one of the following equalities $i_l = i_{l+1}, i_l j_{l+1}, j_l = i_{l+1}, j_l = j_{l+1}$ holds for all $l = 0, 1, \ldots, N+1$. The relationship between the Lie bracket operation on $\mathcal{B}$ and the product operation on S_n gives $\prod_{l=N+1}^{1} \mathrm{ad}_{\Omega_{i_l,j_l}}(\Omega_{i_0,j_0}) = \Omega_{ab}$ or $-\Omega_{ab}$. Therefore, we can generate any $\Omega_{ij} \in \mathcal{B}$ by the iterated Lie brackets of elements in $\mathcal{S}$, and so $\mathrm{Lie}(\mathcal{S}) = \mathfrak{so}(n)$, which implies that the system (8) is controllable.

□

Theorem 1 provides not only an alternative approach to effectively examine the controllability of systems defined on SO(n) through the notion of permutation group, but also a systematic procedure to characterize the controllable submanifold when the system is not fully controllable.

Corollary 1. *The controllable submanifold of the system in* (8) *is uniquely determined by* $\iota(\mathcal{S})$*, where* $\mathcal{S} \subseteq \{\Omega_{i_1,j_1}, \ldots, \Omega_{i_m,j_m}\}$.

Proof. Because the map ι is surjective by Lemma 3, it suffices to show that every permutation determines a subgroup of SO(n). Let $\sigma \in S_n$ be a cycle, say $\sigma = (a_1, \ldots, a_k)$, then there exists some $\mathcal{F} \in \mathcal{P}(\mathcal{B})$ such that $\iota(\mathcal{F}) = \sigma$ by the subjectivity of ι. Thus, every Ω_{a_i,a_j} can be generated by the iterated Lie brackets of elements in $\mathcal{F}$ through the procedure in the proof of Theorem 1, which gives $\mathrm{Lie}(\mathcal{F}) = \mathrm{span}\{\Omega_{a_i,a_j} : 1 \leq i, j \leq k \text{ and } a_i < a_j\}$. As a Lie subalgebra, $\mathrm{Lie}(\mathcal{F})$ is closed under the Lie bracket operation, so $\mathrm{Lie}(\mathcal{F})$ determines an involutive distribution on SO(n). Applying the Frobenius Theorem, $\mathrm{Lie}(\mathcal{F})$ is complete integrable. Let G be the connected Lie subgroup of SO(n) whose Lie algebra is $\mathrm{Lie}(\mathcal{F})$, then by the Lie correspondence Theorem [14], the integral manifolds of $\mathrm{Lie}(\mathcal{F})$ are exactly $GX(0)$, where $X(0)$ is the initial condition of the system (8). In addition, the collection of submanifolds $\{GX(0)\}_{X(0)\in \mathrm{SO}(n)}$ forms a foliation of SO(n). □

C. Controllability of Ensemble Systems on SO(n)

In this subsection, we will use the tools developed in Sections II-B and III to analyze ensemble controllability of systems defined on SO(n).

Theorem 2. *Consider an ensemble of systems evolving on* SO(n)*, given by*

$$\frac{d}{dt}X(t,\varepsilon) = \left[\sum_{k=1}^{m} \varepsilon_k u_k(t)\, \Omega_{i_k,j_k}\right] X(t,\varepsilon), \quad X(0,\varepsilon) = I, \tag{9}$$

where $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_m)' \in K \subset (\mathbb{R}^+)^m$*,* K *is compact,* $X(t,\cdot) \in C(K, \mathrm{SO}(n))$*,* $u_k(t) \in \mathbb{R}$ *for all* $k = 1, \ldots, m$*, and* $m \geq n-1$*. Then, the ensemble systems in* (9) *is ensemble controllable on* $C(K, \mathrm{SO}(n))$ *if and only if there is a subset* $\mathcal{S}$ *of* $\{\Omega_{i_1,j_1}, \ldots, \Omega_{i_m,j_m}\}$ *such that* $\iota(\mathcal{S})$ *is a cycle of length* n.

Proof. (Necessity): If the system in (9) is ensemble controllable, then each single system corresponding to a fixed $\varepsilon \in K$ is also controllable. By the Theorem 1, there is a subset $\mathcal{S}$ of $\{\Omega_{i_1,j_1}, \ldots, \Omega_{i_m,j_m}\}$ such that $\iota(\mathcal{S})$ is a cycle of length n.

(Sufficiency): If there is a subset $\mathcal{S}$ of $\mathcal{F} = \{\Omega_{i_1,j_1}, \ldots, \Omega_{i_m,j_m}\}$ such that $\iota(\mathcal{S})$ is a cycle of length n, then for any $\Omega_{ij} \in \mathcal{B}\backslash\mathcal{F}$, it can be generated by iterated Lie brackets of the elements in $\mathcal{S}$. Applying the generating procedure as presented in the proof of the Theorem 1 to the set $\varepsilon\mathcal{F} = \{\varepsilon_1\Omega_{i_1,j_1}, \ldots, \varepsilon_m\Omega_{i_m,j_m}\}$, we can produce $p(\varepsilon)\Omega_{ij}, q(\varepsilon)\Omega_{i+1,j}, r(\varepsilon)\Omega_{i,i+1}$ for some nonnegative monomials p, q, r defined on K. Let $\mathcal{R}_p, \mathcal{R}_q, \mathcal{R}_r$ denote the ranges of p, q, r, respectively, then by the continuity of monomial functions, these are compact subsets of $\mathbb{R}^+$, and so is their cartesian product $\mathcal{R} = \mathcal{R}_p \times \mathcal{R}_q \times \mathcal{R}_r$ by the Tychonoff's product theorem [15]. Now, we can consider the following reduced system characterized by p, q, r,

$$\begin{aligned}\frac{d}{dt}X(t,\eta) =& \left[v_1(t)\eta_1\Omega_{ij} + v_2(t)\eta_2\Omega_{i+1,j} + v_3(t)\eta_3\Omega_{i,i+1}\right]\cdot \\ & X(t,\eta),\end{aligned} \tag{10}$$

where $\eta = (\eta_1, \eta_2, \eta_3)' = (p(\varepsilon), q(\varepsilon), r(\varepsilon))' \in \mathcal{R}$, $X(t,\eta) \in \mathrm{SO}(n)$, and $v_1(t), v_2(t), v_3(t) \in \mathbb{R}$ are controls. Notice that $\{\Omega_{ij}, \Omega_{i+1,j}, \Omega_{i,i+1}\}$ forms a Lie subalgebra of $\mathfrak{so}(n)$, which is isomorphic to $\mathfrak{so}(3)$, and, therefore, the system in (10) is an ensemble on the space $C(\mathcal{R}, \mathrm{SO}(3))$. In Theorem 1, we have shown that an ensemble system on SO(3) with three controls and three parameter variations are uniformly ensemble controllable on the space of continuous SO(3)-valued functions defined on a compact subset of $(\mathbb{R}^+)^3$, so the system in (10) is ensemble controllable with respect to the parameters η_k for $k = 1, 2, 3$, and its orbit manifold M_{ij} containing the identity matrix is a subgroup of $C(K, \mathrm{SO}(n))$ which is isomorphic to $C(\mathcal{R}, \mathrm{SO}(3))$. Furthermore, one can show that the set $\{M_{ij} : 1 \leq i, j \leq n\}$ forms a cover of $C(K, \mathrm{SO}(n))$. Since $\Omega_{ij} \in \mathcal{B}$ is arbitrary, we have ensemble controllability for the system in (9). □

A direct result can be derived following Theorem 2.

Corollary 2. *The ensemble system as in* (9) *can not be controllable if the number of controls* m *is less than* $n-1$.

Proof. A cycle of length n cannot be decomposed as a product of transpositions with less than $n-1$ terms. □

Remark 2. *An ensemble of systems on* $\mathrm{SO}(2)$ *cannot be ensemble controllable. Because* $\mathfrak{so}(2)$ *is a one dimensional real vector space with the basis* Ω_{12}*, an ensemble of systems on* $\mathrm{SO}(2)$ *has the unique form*

$$\frac{d}{dt}X(t,\varepsilon)=\varepsilon u(t)\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}X(t,\varepsilon). \qquad (11)$$

However, $\mathfrak{so}(2)$ *is nilpotent, so we cannot generate terms as* $\varepsilon^k\Omega_{ij}$ *for* $k\geq 2$ *through iterated Lie brackets. As a result,* $\mathrm{Lie}(\varepsilon\Omega_{12})$ *is not a vector space over* $C(K,\mathbb{R})$*, which implies that the system in* (11) *is not ensemble controllable.*

IV. CONCLUSION

In this paper, we study the control of time-invariant bilinear ensemble systems defined on the special orthogonal group parameterized by a vector of dispersion parameters, which takes values over a positive, compact or locally compact set. We construct an algebraic criterion using the theory of symmetric groups to examine controllability as well as to characterize the controllable submanifold of an ensemble system on SO(n). We also show that controllability of each individual subsystem in the ensemble infers controllability of the whole ensemble for the systems on SO(n). Comparing with the classical Lie algebra rank condition, our approach offers a transparent and efficient verification for determining controllability or identifying controllable submanifolds for uncontrollable systems. The established framework is immediately applicable and extendable to study broader classes of ensemble systems on Lie groups, e.g., on the special Euclidean group SE(n) that is closely related to SO(n).

V. APPENDIX

Theorem 3. *Let* G *be a compact connected Lie group and* $\mathfrak{g}$ *be its Lie algebra, then a driftless bilinear system on* G *of the form*

$$\frac{d}{dt}X(t)=\left(\sum_{i=1}^{m}u_i(t)B_i\right)X(t),$$

where $X(t)\in G$*,* $B_i\in\mathfrak{g}$ *and* $u_i(t)\in\mathbb{R}$*, is controllable if and only if* $\mathrm{Lie}\{B_1,\ldots,B_m\}=\mathfrak{g}$*.*

Proof. See [9] and [11]. □

Theorem 4. *Let* X *be a locally compact Hausdorff space. If* $\mathcal{A}$ *is a closed subalgebra of* $C_0(X,\mathcal{R})$ *that separates points, then either* $\mathcal{A}=C_0(X,\mathbb{R})$ *or* $\mathcal{A}=\{f\in C_0(X,\mathbb{R}): f(x_0)=0\}$ *for some* $x_0\in X$*, where* $C_0(X,\mathbb{R})=\{f\in C(X,\mathbb{R}): f$ *vanishes at infty*$\}$*.*

Proof. See [16]. □

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