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A toolkit for nonlinear feedback design *

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Abstract Motivated by several recent results, we assemble a set of basic tools which can be used to construct systematic procedures for nonlinear feedback design As an illustration, we construct a backstepping procedure for observer-based global stabilization and tracking of a class of nonlinear systems

Keywords Design tools, nonlinear damping, integrator backstepping, observer-based design, global tracking

1. Introduction

The complexity of the nonlinear output-feedback problem challenges not only the researcher's knowledge of nonlinear geometric techniques [4,16], but also his/her ability to employ, and often invent, a wide variety of other tools. This is particularly apparent in several recent results which make use of intricate combinations of diverse concepts: special classes of systems characterized by geometric conditions [8,9,15] or 'control Lyapunov functions' [17,5], strict positive real properties of some part of the system [10,12] and filtered transformations to guarantee these properties [13,15], means to deal with swapping terms [9] or to avoid them altogether [15], etc. Proofs combining these tools may appear too technical and discourage potential users.

A more systematic treatment, which supplements rigor with intuitive appeal, seems to be needed, and we make a step in this direction. In Section 2 we assemble a set of four simple tools for nonlinear feedback design, either with or without full-state measurement. The first two of these tools, 'nonlinear damping' and 'integrator backstepping', were used previously in adaptive and nonlinear control [3,1,22,2,10,8,14,15,5]. In Section 3 the tools of Section 2 are employed to give an alternative solution to an output-feedback problem recently solved by Marino and Tomei [15]. The dynamic part of the controller designed in Section 3 consists of only a nonlinear observer, while in [15] it also contains the filters required for the filtered transformations.

2. The design toolkit

Throughout this section it is assumed that a feedback control $u = \alpha(x)$ is known, which, when applied to the system

$$\dot{x} = f(x) + g(x)u, \quad x \in \mathbb{R}^n, \; u \in \mathbb{R}, \tag{2.1}$$

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guarantees global boundedness of $x(t)$ and regulation of $\eta(x) \in \mathbb{R}^m$, that is, $\eta(x(t)) \to 0$ as $t \to \infty$. These properties are ascertained by the inequality

$$\frac{\partial V}{\partial x}(x)[f(x)+g(x)\alpha(x)] \leq -W(\eta(x)) \leq 0, \quad \forall x \in \mathbb{R}^n, \tag{2.2}$$

where $V(x)$ is positive definite and radially unbounded, and $W(\eta)$ is positive definite. It is further assumed that f, g, η, α, V and W are C^2 on $\mathbb{R}^n$.

Each of the four lemmas in this section employs $V(x)$ and $\alpha(x)$ to design a feedback control for a particular perturbed or augmented version of (2.1).

In the first lemma, the system (2.1) is perturbed by an unknown disturbance. As the following example demonstrates, even an exponentially decaying disturbance may cause explosive forms of unbounded behavior if it multiplies a term with significant nonlinear growth rate.

Example 1. Suppose that in the system

$$\dot{x} = x^3 + u + x^2 k\, \mathrm{e}^{-t}, \tag{2.3}$$

the term $x^2 k\, \mathrm{e}^{-t}$, where k is an unknown constant, is considered as a perturbation. If, because of the factor e^{-t}, this perturbation is neglected and the control for the unperturbed system $u = -x - x^3$ is applied to the perturbed system (2.3), the resulting feedback system is

$$\dot{x} = -x + x^2 k\, \mathrm{e}^{-t}. \tag{2.4}$$

Since (2.4) is linear in $1/x$, its explicit solution is known:

$$x(t) = \frac{2x(0)}{(2-kx(0))\, \mathrm{e}^{t} + kx(0)\, \mathrm{e}^{-t}} \tag{2.5}$$

The denominator of (2.5) is zero at $t = \frac{1}{2}\ln[kx(0)/(kx(0)-2)]$. It follows that, whenever $kx(0) > 2$, $x(t)$ escapes to infinity in finite time.

It is clear from this example that additional control action is needed to attenuate the effects of the unknown disturbance. In the following lemma, a 'nonlinear damping' term is designed to fullfill this task for a class of unknown disturbances which are in the span of the control.

Lemma NDM (Nonlinear Damping – Matched). *Consider the perturbed system*

$$\dot{x} = f(x) + g(x)\left[u + p(x)^{\mathrm{T}} d(x, \varepsilon)\right], \tag{2.6}$$

where $p(x)$, $d(x, \varepsilon)$ are continuous and $d(x, 0) \equiv 0$. Let the 'disturbance generator'

$$\dot{\varepsilon} = q(x, \varepsilon), \quad q(x, 0) \equiv 0, \quad \varepsilon \in \mathbb{R}^l, \tag{2.7}$$

satisfy the inequality

$$\frac{\partial \Omega}{\partial \varepsilon}(\varepsilon) q(x, \varepsilon) \leq -\| d(x, \varepsilon) \|^2 \tag{2.8}$$

for some positive definite radially unbounded function $\Omega(\varepsilon)$ and for all $\varepsilon \in \mathbb{R}^l$, $x \in \mathbb{R}^n$. Then, the feedback control

$$u = \alpha(x) - \frac{\partial V}{\partial x}(x) g(x) \| p(x) \|^2 \triangleq \alpha_{\mathrm{NDM}}(x), \tag{2.9}$$

when applied to (2.6), guarantees global boundedness of $x(t)$ and regulation of $\eta(x)$.

Proof. Because of (2.2) and (2.8), the time derivative of $V_{\mathrm{NDM}}(x, \varepsilon) = V(x) + \Omega(\varepsilon)$ for (2.7) and the perturbed system (2.6) with the feedback (2.9) is

$$\begin{aligned}\dot{V}_{\mathrm{NDM}}(x, \varepsilon) &= \frac{\partial V}{\partial x}[f + g\alpha] - \left[\frac{\partial V}{\partial x} g\right]^2 \| p \|^2 + \frac{\partial V}{\partial x} g p^{\mathrm{T}} d + \frac{\partial \Omega}{\partial \varepsilon} q \\ &\le -W(\eta) - \left[\frac{\partial V}{\partial x} g\right]^2 \| p \|^2 + \frac{\partial V}{\partial x} g p^{\mathrm{T}} d - \| d \|^2 \\ &\le -W(\eta) - \tfrac{3}{4} \| d \|^2 - \left\| \tfrac{1}{2} d - \frac{\partial V}{\partial x} g p \right\|^2 \\ &\le -W(\eta) - \tfrac{3}{4} \| d \|^2, \end{aligned} \tag{2.10}$$

which proves global boundedness of $x(t)$, $\varepsilon(t)$. Furthermore, LaSalle's invariance theorem quarantees that $x(t)$, $\varepsilon(t)$ converge to the largest invariant set of (2.6)–(2.7) on which $\dot{V}_{\mathrm{NDM}}(x, \varepsilon) = 0$. This proves that the regulation of $\eta(x)$ is achieved and that the disturbance vanishes: $d(x(t), \varepsilon(t)) \to 0$ as $t \to \infty$. □

The control $\alpha_{\mathrm{NDM}}(x)$ in (2.9) is designed by adding a term to the control $\alpha(x)$ for the unperturbed system. The implementation of this nonlinear damping term does not require that $d(x, \varepsilon)$, $q(x, \varepsilon)$ or $\Omega(\varepsilon)$ be known: it is sufficient that they satisfy (2.8).

The nonlinear damping (2.9) is a variant of a design by Barmish, Corless and Leitmann [1]. Its effectiveness as a tool for output-feedback design was suggested by Sontag [20,21] and demonstrated by Marino and Tomei [15]. A form of nonlinear damping is implicit in an early adaptive control result by Feuer and Morse [3].

Example 1 (continued). Using $p(x) = x^2$, $d(x, \varepsilon) = \varepsilon$, and the disturbance generator $\dot{\varepsilon} = -\varepsilon$, $\varepsilon(0) = k$, the perturbed system (2.3) is rewritten in the form (2.6) as

$$\dot{x} = x^3 + u + x^2 \varepsilon. \tag{2.11}$$

With $\alpha(x) = -x - x^3$, $V(x) = \frac{1}{2}x^2$ and $\Omega(\varepsilon) = \frac{1}{2}\varepsilon^2$, Lemma NDM applies and the nonlinear damping feedback is

$$u = \alpha_{\mathrm{NDM}}(x) = -x - x^3 - x^5. \tag{2.12}$$

With $V_{\mathrm{NDM}} = \frac{1}{2}(x^2 + \varepsilon^2)$ it is easy to show that the resulting closed-loop system

$$\dot{x} = -x + x^2 k \, \mathrm{e}^{-t} - x^5 \tag{2.13}$$

is globally exponentially stable. Clearly, the nonlinear damping term $-x^5$ prevented the unbounded behavior that would have been caused by $x^2 k \, \mathrm{e}^{-t}$.

In the second lemma, 'integrator backstepping' is used to design a feedback control for the system obtained when (2.1) is augmented by an integrator.

Lemma IB (Integrator Backstepping). *Consider the augmented system*

$$\dot{x} = f(x) + g(x)\xi, \tag{2.14a}$$

$$\dot{\xi} = u, \tag{2.14b}$$

where $\xi \in \mathbb{R}$ is available for measurement. Then, the feedback control

$$u = -(\xi - \alpha(x)) + \frac{\partial \alpha}{\partial x}(x)[f(x) + g(x)\xi] - \frac{\partial V}{\partial x}(x) g(x) \triangleq \alpha_{\mathrm{IB}}(x, \xi), \tag{2.15}$$

when applied to the system (2.14), *guarantees global boundedness of* $x(t)$, $\xi(t)$ *and regulation of* $\eta(x)$, $\xi-\alpha(x)$.

Proof. The backstepping idea is to first view ξ as the control and stabilize (2.14a) with $\alpha(x)$ from (2.2). Then, to account for the fact that ξ is not the control, the change of variables $z=\xi-\alpha(x)$ is introduced to transform (2.14) into

$$\dot{x}=f(x)+g(x)(\alpha(x)+z), \tag{2.16a}$$

$$\dot{z}=u-\frac{\partial\alpha}{\partial x}(x)\left[f(x)+g(x)(\alpha(x)+z)\right]. \tag{2.16b}$$

Finally, (2.15) is designed to make the time derivative of $V_{IB}(x, z)=V(x)+\frac{1}{2}z^2$ nonpositive:

$$\dot{V}_{IB}(x, z)=\frac{\partial V}{\partial x}(f+g\alpha)+\frac{\partial V}{\partial x}gz-z^2-\frac{\partial V}{\partial x}gz\leq -W(\eta)-z^2. \tag{2.17}$$

This proves global boundedness of $x(t)$, $z(t)$ and, by LaSalle's invariance theorem, regulation of $\eta(x)$ and z. □

Integrator backstepping has recently been used by several authors [22,2,10], and was one of the tools for the systematic design of adaptive nonlinear controllers by Kanellakopoulos, Kokotovic and Morse 8] and Jiang and Praly [5]. As a design tool, it often leads to global results not achievable by feedback linearization, as illustrated by the following example.

Example 2. Consider a system which is not globally feedback linearizable:

$$\dot{x}=x^2+(1+x)\xi, \tag{2.18a}$$

$$\dot{\xi}=u. \tag{2.18b}$$

Thinking of (2.18a) as a system controlled by ξ, we use $\alpha(x)=-x$, $V(x)=\frac{1}{2}x^2$ to satisfy (2.2). Recognizing that ξ is not the control, we let $z=\xi+x$ and transform (2.18) into

$$\dot{x}=-x+(1+x)z, \tag{2.19a}$$

$$\dot{z}=u-x+(1+x)z. \tag{2.19b}$$

For this system, $\alpha_{IB}(x, \xi)$ given by (2.15) is

$$u=-\xi-(\xi+2x)(1+x)=-z+x-z(1+x)-x(1+x). \tag{2.20}$$

With $V_{IB}(x, z)=\frac{1}{2}(x^2+z^2)$ it is easy to show that the system (2.19) controlled by (2.20) is globally exponentially stable.

Lemmas NDM and IB can be combined into more sophisticated tools. One such combination, incorporating filtered transformations, was used by Marino and Tomei in [15]. Two additional combinations are given in Lemmas NDE and OIB below.

While in Lemma NDM the perturbation $p(x)^{T}d(x, \varepsilon)$ was in the span of the control u, in Lemma NDE it precedes the control by one integrator.

Lemma NDE (Nonlinear Damping – Extended). *For the augmented perturbed system*

$$\dot{x}=f(x)+g(x)\left[\xi+p(x)^{T}d(x, \varepsilon)\right], \tag{2.21a}$$

$$\dot{\xi}=u, \tag{2.21b}$$

under the assumptions of Lemma NDM, *the feedback control*

$$u = -[\xi - \alpha_{\mathrm{NDM}}(x)] + \frac{\partial \alpha_{\mathrm{NDM}}}{\partial x}(x)[f(x) + g(x)\xi] - \frac{\partial V}{\partial x}(x) g(x)$$
$$- [\xi - \alpha_{\mathrm{NDM}}(x)] \left\| \frac{\partial \alpha_{\mathrm{NDM}}}{\partial x}(x) g(x) p(x) \right\|^2 \triangleq \alpha_{\mathrm{NDE}}(x, \xi) \tag{2.22}$$

guarantees global boundedness of $x(t)$, $\xi(t)$ *and regulation of* $\eta(x)$, $\xi - \alpha_{\mathrm{NDM}}(x)$.

Proof. Viewing ξ as the control, the nonlinear damping feedback of Lemma NDM for (2.21a) is $\xi = \alpha_{\mathrm{NDM}}(x)$, with $V_{\mathrm{NDM}}(x, \varepsilon) = V(x) + \Omega(\varepsilon)$. As in Lemma IB, $z = \xi - \alpha_{\mathrm{NDM}}(x)$ is used to transform (2.21) into

$$\dot{x} = f(x) + g(x)\alpha_{\mathrm{NDM}}(x) + g(x)p(x)^{\mathrm{T}} d(x, \varepsilon) + g(x)z, \tag{2.23a}$$

$$\dot{z} = u - \frac{\partial \alpha_{\mathrm{NDM}}}{\partial x}(x)[f(x) + g(x)\alpha_{\mathrm{NDM}}(x) + g(x)z] - \bar{p}(x)^{\mathrm{T}} d(x, \varepsilon), \tag{2.23b}$$

where $\bar{p}(x) \triangleq (\partial \alpha_{\mathrm{NDM}}(x)/\partial x) g(x) p(x)$. In the absence of the term $\bar{p}(x)^{\mathrm{T}} d(x, \varepsilon)$, Lemma IB would result in the feedback control

$$u = -z + \frac{\partial \alpha_{\mathrm{NDM}}}{\partial x}[f + g\alpha_{\mathrm{NDM}} + gz] - \frac{\partial V}{\partial x} g \triangleq \bar{\alpha}_{\mathrm{IB}}(x, z) \tag{2.24}$$

with $\bar{V}_{\mathrm{IB}}(x, z, \varepsilon) = V_{\mathrm{NDM}}(x, \varepsilon) + \frac{1}{2}z^2$. To account for the presence of $\bar{p}(x)^{\mathrm{T}} d(x, \varepsilon)$, we apply Lemma NDM once again and add a nonlinear damping term to (2.24):

$$u = \bar{\alpha}_{\mathrm{IB}}(x, z) - \frac{\partial \bar{V}_{\mathrm{IB}}}{\partial z} \| \bar{p}(x) \|^2 = \bar{\alpha}_{\mathrm{IB}}(x, z) - z \left\| \frac{\partial \alpha_{\mathrm{NDM}}}{\partial x}(x) g(x) p(x) \right\|^2. \tag{2.25}$$

When applied to (2.23), this control guarantees global boundedness of $x(t)$, $z(t)$ and regulation of $\eta(x)$ and z. □

The tools we presented so far assumed full-state measurement. Suppose now that the system (2.1) is augmented by an integrator whose state is not measured, but is instead estimated by an observer. We consider (2.14) for the case $g(x) \equiv g \neq 0$:

$$\dot{x} = f(x) + g\xi, \tag{2.26a}$$

$$\dot{\xi} = u. \tag{2.26b}$$

Following [11], an observer for this system is

$$\dot{\hat{y}} = -k_1(\hat{y} - y) + g_i \hat{\xi} + f_i(x), \tag{2.27a}$$

$$\dot{\hat{\xi}} = -k_2(\hat{y} - y) + u, \tag{2.27b}$$

where $y = x_i$ is a component of x such that $g_i \neq 0$ and k_1, k_2 are chosen to guarantee the exponential stability of the error system

$$\begin{bmatrix} \dot{\tilde{y}} \\ \dot{\tilde{\xi}} \end{bmatrix} = \begin{bmatrix} -k_1 & g_i \\ -k_2 & 0 \end{bmatrix} \begin{bmatrix} \tilde{y} \\ \tilde{\xi} \end{bmatrix} \triangleq A_0 \begin{bmatrix} \tilde{y} \\ \tilde{\xi} \end{bmatrix}, \tag{2.28}$$

where $\tilde{y} = y - \hat{y}$, $\tilde{\xi} = \xi - \hat{\xi}$. Then, an observer-based feedback control for (2.26) is designed by backstepping the integrator (2.27b) in the observer.

Lemma OIB (Observed-Integrator Backstepping). *Consider the augmented system* (2.26), *where the unmeasured state* ξ *is estimated by the observer* (2.27). *Then, with* $\alpha_1(x) \triangleq \alpha(x) - (\partial V(x)/\partial x)g$, *the feedback control*

$$u = k_2(y-\hat{y}) - \left[\hat{\xi} - \alpha_1(x)\right] + \frac{\partial \alpha_1}{\partial x}(x)\left[f(x) + g\hat{\xi}\right] - \frac{\partial V}{\partial x}(x)g - \left[\hat{\xi} - \alpha_1(x)\right]\left[\frac{\partial \alpha_1}{\partial x}(x)g\right]^2, \tag{2.29}$$

when applied to the system (2.26), *guarantees global boundedness of* $x(t)$, $\xi(t)$ *and regulation of* $\eta(x)$, $\xi - \alpha_1(x)$.

Proof. Let us combine (2.26a) with the observer equation (2.27b) into a system:

$$\dot{x} = f(x) + g\left[\hat{\xi} + \tilde{\xi}\right], \tag{2.30a}$$

$$\dot{\hat{\xi}} = -k_2(y-\hat{y}) + u, \tag{2.30b}$$

and treat $\tilde{\xi}$ as a disturbance generated by (2.28). The system (2.30) is in the form (2.21) with $\varepsilon = [\tilde{y}\ \tilde{\xi}]^{\mathrm{T}}$, $d(x, \varepsilon) = \tilde{\xi}$, $p(x) = 1$ and $\Omega(\varepsilon) = \varepsilon^{\mathrm{T}} P_0 \varepsilon$, $P_0 A_0 + A_0^{\mathrm{T}} P_0 = -I$. Hence, Lemma NDE applies and the feedback control (2.29) guarantees global boundedness of $x(t)$, $\hat{\xi}(t)$, $\varepsilon(t)$ and regulation of $\eta(x)$, $\hat{\xi} - \alpha_1(x)$. Then, global boundedness of $\xi(t)$ and regulation of $\xi - \alpha_1(x)$ follow from $\xi = \hat{\xi} + \tilde{\xi} = z + \alpha_1(x) + \tilde{\xi}$ and $\tilde{\xi} \to 0$. □

3. A backstepping design procedure

Employing the tools of Section 2 in a step-by-step fashion, we are able to construct backstepping procedures for nonlinear feedback design problems. With full-state feedback, such procedures have been constructed for partially linear composite systems in [18] and for a class of nonlinear systems containing unknown constant parameters in [8]. Here we design an observer-based controller for the class of nonlinear systems that can be transformed via a global diffeomorphism into the output-feedback form

$$\dot{\zeta} = A\zeta + \phi(y) + b\sigma(y)u, \tag{3.1a}$$

$$y = c^{\mathrm{T}}\zeta = \zeta_1, \tag{3.1b}$$

$$A = \begin{bmatrix} 0 & & I \\ \vdots & & \\ 0 & \cdots & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ b_{n-\rho} \\ \vdots \\ b_0 \end{bmatrix}, \quad c^{\mathrm{T}} = [1\ \ 0\ \cdots\ 0], \quad \phi(y) = \begin{bmatrix} \phi_1(y) \\ \vdots \\ \phi_n(y) \end{bmatrix}, \tag{3.2}$$

where only y is available for measurement, $b_{n-\rho}s^{n-\rho} + \cdots + b_1 s + b_0$ is a Hurwitz polynomial, and $\phi_1, \ldots, \phi_n, \sigma$ are smooth functions with $\sigma(y) \neq 0\ \forall y \in \mathbb{R}$. This class of nonlinear systems, characterized via geometric conditions in [15, Theorem 5.1], is of interest because its state can be estimated using the observer

$$\dot{\hat{\zeta}} = A\hat{\zeta} + K_0(\hat{\zeta}_1 - y) + \phi(y) + b\sigma(y)u, \tag{3.3}$$

where K_0 is chosen so that $A_0 = A - K_0 c^{\mathrm{T}}$ in the error system $\dot{\tilde{\zeta}} = A_0\tilde{\zeta}$, $\tilde{\zeta} = \zeta - \hat{\zeta}$, is Hurwitz. Using this observer, we now design a feedback controller to force the output y of (3.1) to track a reference signal $y_r(t)$, which is given along with its first ρ derivatives $\dot{y}_r, \ddot{y}_r, \ldots, y_r^{(\rho)}$.

In Step 1, our step-by-step design applies Lemma OIB to the first two equations of (3.1). Each consecutive step adds one more equation from (3.3) and applies Lemmas IB and NDM. The procedure terminates at Step $\rho - 1$, when the true control appears for the first time.

Step 1: We define the output error $z_1 = y - y_r \triangleq y - \alpha_0(y_r)$ and consider the second-order system

$$\dot{z}_1 = \zeta_2 + \phi_1(y) - \dot{y}_r \triangleq \zeta_2 + \beta_1(y, y_r, \dot{y}_r), \tag{3.4a}$$

$$\dot{\zeta}_2 = \zeta_3 + \phi_2(y). \tag{3.4b}$$

As in Lemma OIB, we replace (3.4b) with the second equation of the observer (3.3) to compose the system

$$\dot{z}_1 = \hat{\zeta}_2 + \beta_1(y, y_r, \dot{y}_r) + \tilde{\zeta}_2, \tag{3.5a}$$

$$\dot{\hat{\zeta}}_2 = \hat{\zeta}_3 + K_{02}(\hat{\zeta}_1 - y) + \phi_2(y). \tag{3.5b}$$

If $\hat{\zeta}_2$ were the control, then Lemma NDM would result in

$$\hat{\zeta}_2 = -\beta_1(y, y_r, \dot{y}_r) - 2z_1 \triangleq \alpha_1(y, y_r, \dot{y}_r) \tag{3.6}$$

and $V_1(z_1, \tilde{\zeta}) = \frac{1}{2}z_1^2 + \tilde{\zeta}^T P_0 \tilde{\zeta}$, where $P_0 A_0 + A_0^T P_0 = -I$. Since $\hat{\zeta}_2$ is not the control, the new state $z_2 = \hat{\zeta}_2 - \alpha_1(y, y_r, \dot{y}_r)$ is introduced and (3.5) is rewritten as

$$\dot{z}_1 = -z_1 + z_2 - z_1 + \tilde{\zeta}_2, \tag{3.7a}$$

$$\begin{aligned} \dot{z}_2 &= \hat{\zeta}_3 + K_{02}(\hat{\zeta}_1 - y) + \phi_2(y) - \frac{\partial \alpha_1}{\partial y}(\hat{\zeta}_2 + \phi_1(y) + \tilde{\zeta}_2) - \frac{\partial \alpha_1}{\partial y_r}\dot{y}_r - \frac{\partial \alpha_1}{\partial \dot{y}_r}\ddot{y}_r \\ &\triangleq \hat{\zeta}_3 + \beta_2(y, \hat{\zeta}_1, \hat{\zeta}_2, y_r, \dot{y}_r, \ddot{y}_r) - \frac{\partial \alpha_1}{\partial y}\tilde{\zeta}_2, \end{aligned} \tag{3.7b}$$

where in the definition of β_2 we have used the fact that $\partial\alpha_1/\partial y$, $\partial\alpha_1/\partial y_r$, $\partial\alpha_1/\partial \dot{y}_r$ are known functions of y, y_r and $\dot{y}_r$. Again, if $\hat{\zeta}_3$ were the control, then Lemmas IB and NDM would result in

$$\hat{\zeta}_3 = -\beta_2(y, \hat{\zeta}_1, \hat{\zeta}_2, y_r, \dot{y}_r, \ddot{y}_r) - \left[1 + \left(\frac{\partial \alpha_1}{\partial y}\right)^2\right] z_2 - z_1 \triangleq \alpha_2(y, \hat{\zeta}_1, \hat{\zeta}_2, y_r, \dot{y}_r, \ddot{y}_r), \tag{3.8}$$

and $V_2(z_1, z_2, \tilde{\zeta}) = V_1(z_1, \tilde{\zeta}) + \frac{1}{2}z_2^2 + \tilde{\zeta}^T P_0 \tilde{\zeta}$.

Step i ($2 \le i < \rho - 1$): In Steps 2 through $i - 1$ we designed $\alpha_1, \ldots, \alpha_i$. Now, in Step i, we apply Lemma IB to backstep another integrator. We introduce the new state $z_{i+1} = \hat{\zeta}_{i+1} - \alpha_i(y, \hat{\zeta}_1, \ldots, \hat{\zeta}_i, y_r, \dot{y}_r, \ldots, y_r^{(i)})$ to obtain

$$\dot{z}_j = -z_j + z_{j+1} - \frac{\partial \alpha_{j-1}}{\partial y}\tilde{\zeta}_2 - \left(\frac{\partial \alpha_{j-1}}{\partial y}\right)^2 z_j - z_{j-1}, \quad 1 \le j \le i, \tag{3.9a}$$

$$\dot{z}_{i+1} = \hat{\zeta}_{i+2} + \beta_{i+1}(y, \hat{\zeta}_1, \ldots, \hat{\zeta}_{i+1}, y_r, \dot{y}_r, \ldots, y_r^{(i+1)}) - \frac{\partial \alpha_i}{\partial y}\tilde{\zeta}_2. \tag{3.9b}$$

Again, if $\hat{\zeta}_{i+2}$ were the control, Lemmas IB and NDM would result in

$$\begin{aligned} \hat{\zeta}_{i+2} &= -\beta_{i+1}(y, \hat{\zeta}_1, \ldots, \hat{\zeta}_{i+1}, y_r, \dot{y}_r, \ldots, y_r^{(i+1)}) - \left[1 + \left(\frac{\partial \alpha_i}{\partial y}\right)^2\right] z_{i+1} - z_i \\ &\triangleq \alpha_{i+1}(y, \hat{\zeta}_1, \ldots, \hat{\zeta}_{i+1}, y_r, \dot{y}_r, \ldots, y_r^{(i+1)}) \end{aligned} \tag{3.10}$$

and $V_{i+1} = V_i + \frac{1}{2}z_{i+1}^2 + \tilde{\zeta}^T P_0 \tilde{\zeta}$.

Step $\rho - 1$: Finally, we backstep the last integrator before the actual control u appears. Following Lemma IB, we substitute $z_\rho = \hat{\zeta}_\rho - \alpha_{\rho-1}(y, \hat{\zeta}_1, \ldots, \hat{\zeta}_{\rho-1}, y_r, \dot{y}_r, \ldots, y_r^{(\rho-1)})$ into (3.9) (with $i = \rho - 2$) and augment the resulting system with the $\dot{z}_\rho$-equation:

$$\dot{z}_j = -z_j + z_{j+1} - \frac{\partial \alpha_{j-1}}{\partial y}\tilde{\zeta}_2 - \left(\frac{\partial \alpha_{j-1}}{\partial y}\right)^2 z_j - z_{j-1}, \quad 1 \le j \le \rho - 1, \tag{3.11a}$$

$$\dot{z}_\rho = b_{n-\rho}\sigma(y)u + \hat{\zeta}_{\rho+1} + \beta_\rho\left(y, \hat{\zeta}_1, \ldots, \hat{\zeta}_\rho, y_r, \dot{y}_r, \ldots, y_r^{(\rho)}\right) - \frac{\partial \alpha_{\rho-1}}{\partial y}\tilde{\zeta}_2. \tag{3.11b}$$

Now the actual control u has appeared, and Lemmas IB and NDM result in the control law

$$u = -\frac{1}{b_{n-\rho}\sigma(y)}\left\{\hat{\zeta}_{\rho+1} + \beta_\rho\left(y, \hat{\zeta}_1, \ldots, \hat{\zeta}_\rho, y_r, \dot{y}_r, \ldots, y_r^{(\rho)}\right) + \left[1 + \left(\frac{\partial \alpha_{\rho-1}}{\partial y}\right)^2\right] z_\rho + z_{\rho-1}\right\}. \tag{3.12}$$

The derivative of

$$V_\rho = V_{\rho-1} + \tfrac{1}{2}z_\rho^2 + \tilde{\zeta}^{\mathrm{T}}P_0\tilde{\zeta} = \tfrac{1}{2}\sum_{j=1}^{\rho} z_j^2 + \rho\tilde{\zeta}^{\mathrm{T}}P_0\tilde{\zeta} \tag{3.13}$$

along the solutions of (3.11)–(3.12) is nonpositive:

$$\dot{V}_\rho \le -\sum_{j=1}^{\rho}\left[z_j^2 + \tfrac{3}{4}\|\tilde{\zeta}\|^2 + \left(\frac{\partial \alpha_{j-1}}{\partial y}z_j + \tfrac{1}{2}\tilde{\zeta}_2\right)^2\right] \le 0. \tag{3.14}$$

With this systematic procedure we have not only designed the control law (3.12), but have also set the stage for the following result:

Theorem 3.1 (Stability and tracking). *For the nonlinear system* (3.1), *assume that* $b_{n-\rho}s^{n-\rho} + \cdots + b_1 s + b_0$ *is a Hurwitz polynomial, and that* $y_r, \dot{y}_r, \ldots, y_r^{(\rho)}$ *are bounded on* $[0, \infty)$ *and* $y_r^{(\rho)}(t)$ *is piecewise continuous. Then, all the signals in the closed-loop system consisting of the system* (3.1), *the observer* (3.3) *and the control* (3.12) *are globally bounded, and, in addition,*

$$\lim_{t\to\infty}[y(t) - y_r(t)] = 0. \tag{3.15}$$

Proof. Due to the piecewise continuity of $y_r^{(\rho)}(t)$ and the smoothness of the nonlinearities, the solution of the closed-loop system exists. Let its maximum interval of existence be $[0, t_f)$. On this interval, the nonnegative function V_ρ is nonincreasing because of (3.14). Thus, $z_1, \ldots, z_\rho$ are bounded on $[0, t_f)$ by some constants depending only on the initial conditions of (3.1) and (3.3). The boundedness of all other signals on $[0, t_f)$ is established as follows. Since z_1 and y_r are bounded, y is bounded. The boundedness of $\tilde{\zeta}$ and $\tilde{\zeta}_1 = y - \hat{\zeta}_1$ imply that $\hat{\zeta}_1$ is bounded. Since z_2 is bounded, $\hat{\zeta}_2$ is bounded. In the same manner, it can be shown that $\hat{\zeta}_1, \ldots, \hat{\zeta}_\rho$ are bounded. Hence, $\zeta_1, \ldots, \zeta_\rho$ are bounded.

To prove the boundedness of $\zeta_{\rho+1}, \ldots, \zeta_n$, we use the fact (see, for example, [19, Theorem 2.1]) that there exists a similarity transformation $\bar{\zeta} = T\zeta$, with $\bar{\zeta}_1 = \zeta_1, \ldots, \bar{\zeta}_\rho = \zeta_\rho$, which results in

$$\begin{aligned}
\dot{\zeta}_1 &= \zeta_2 + \phi_1(y), \\
&\vdots \\
\dot{\zeta}_{\rho 1} &= \zeta_\rho + \phi_{\rho-1}(y), \\
\dot{\zeta}_\rho &= \phi_\rho(y) + b_{n-\rho}\sigma(y)u + a_1^{\mathrm{T}}\bar{\zeta}, \\
\dot{\bar{\zeta}}^{\mathrm{r}} &= A_{\mathrm{r}}\bar{\zeta}^{\mathrm{r}} + \bar{\phi}(y), \\
y &= \zeta_1,
\end{aligned} \tag{3.16}$$

where the eigenvalues of the $(n-\rho)\times(n-\rho)$ matrix A_r are the roots of the Hurwitz polynomial $b_{n-\rho}s^{n-\rho}+\cdots+b_1 s+b_0$. Now the boundedness of $\bar{\zeta}^r$, which follows from the boundedness of $\bar{\phi}(y)$, together with the boundedness of $\zeta_1,\ldots,\zeta_\rho$, imply that $\bar{\zeta}$ is bounded. We conclude that $\zeta=T^{-1}\bar{\zeta}$ and $\hat{\zeta}=\zeta-\tilde{\zeta}$ are bounded. Since $b_{n-\rho}\sigma(y)$ is bounded away from zero, the feedback control u (3.12) is bounded.

We have thus shown that the state of the closed-loop system is bounded on its maximal interval of existence $[0, t_f)$. Hence, $t_f=\infty$.

To prove the convergence of the tracking error to zero, note that the boundedness of ζ, $\hat{\zeta}$, $\tilde{\zeta}$ and u, together with (3.13) and (3.14) imply that both $\dot{V}_\rho$ and $\ddot{V}_\rho$ are bounded, and, moreover, that $\dot{V}_\rho$ is integrable on $[0,\infty)$. Hence, $\dot{V}_\rho\to 0$ as $t\to\infty$, which proves that $z_1,\ldots, z_\rho\to 0$ as $t\to\infty$. Since $z_1=y-y_r$, this proves (3.15). □

4. Concluding remarks

Among the major contributions of geometric methods to the systematic design of nonlinear feedback systems over the last decade are conditions characterizing classes of nonlinear systems which are feedback linearizable or transformable into so-called 'normal forms'. Under such conditions, the feedback design problem either becomes linear or is greatly simplified due to the special properties of the normal forms.

Geometric results have their own limitations: they are often only locally valid, as in Example 2, or valid only in a disturbance-free setting, which excludes the system of Example 1. Tools like those assembled in this paper alleviate some of these limitations and appear as a valuable supplement to geometric methods. The backstepping procedure of Section 3 demonstrates that such tools are not just 'tricks of the trade', but can also be used in a systematic fashion. In a more complicated adaptive setting, where the backstepping has to be performed not only under state observation, but also under parameter estimation, such tools are currently used to design fundamentally new adaptive schemes.

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