M. Tech. Project Report

Stabilizing a Flexible Beam on a Cart: A Distributed Port Hamiltonian Approach

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by

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Chairman	**Examiner(s)**	**Supervisor**
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Abstract

Motion planning and stabilization of the inverted pendulum on a cart is a much studied problem in the control community. We focus our attention on stabilizing a vertically upright flexible beam fixed on a moving cart. The flexibility of the beam is restricted only to the direction along the traverse of the cart. The control objective is to attenuate the effect of disturbances on the vertically upright profile of the beam. The control action available is the motion of the cart. By regulating this motion, we seek to regulate the shape of the beam. The problem presents a combination of a system described by a partial differential equation (PDE) and a cart modeled as an ordinary differential equation (ODE) as well as controller which we restrict to an ODE. We set our problem in the port controlled Hamiltonian framework. The interconnection of the flexible beam to the cart is viewed as a power conserving interconnection of an infinite dimensional system to a finite dimensional system. The energy Casimir method is employed to obtain the controller. In this method, we look for some constants of motion which are invariant of the choice of controller Hamiltonian. These Casimirs relate the controller states to the states of the system. We finally prove stability of the equilibrium configuration of the closed-loop system.

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List of Symbols

$\langle \bullet \vert \bullet \rangle$	:	Duality product
$\ll \bullet, \bullet \gg$	:	+pairing operator
$\mathcal{F}$	:	Space of flows (Generalized velocities)
$\mathcal{E}$	:	Space of efforts (Generalized forces)
$\mathcal{F} \times \mathcal{E}$	:	Space of power variables
$\mathbb{D}$	:	Dirac Structure
$\wedge$	:	Wedge product of differential forms
$\star$	:	Hodge star operator
d	:	Exterior derivative of differential forms
$T_x\mathcal{M}$	:	Tangent space of a manifold $\mathcal{M}$ at $x \in \mathcal{M}$
$\Omega^k(\mathcal{M})$	:	Space of differential k-forms on a manifold $\mathcal{M}$
l	:	Spatial variable for the flexible beam
$\mathcal{D}$	:	1-dimensional spatial domain for the beam
$\partial\mathcal{D}$	:	0-dimensional boundary of the spatial domain for the beam
$w(l,t)$	:	Deflection of the beam from the equilibrium configuration
$\phi(l,t)$	:	Rotation of the cross-section of the beam
$x(t)$	:	Displacement of the cart in horizontal direction
M	:	Mass of the cart
$\mathcal{H}$	:	Hamiltonian for the system
$\mathcal{H}_{\mathcal{B}}$	:	Hamiltonian for the beam
$\epsilon_t(l,t)$	:	1-form corresponding to translational strain
$\epsilon_r(l,t)$	:	1-form corresponding to rotational strain
$p_t(l,t)$	:	1-form corresponding to translational momentum
$p_r(l,t)$	:	1-form corresponding to rotational momentum
$\Psi^k(\mathcal{D})$	:	Space of k-forms on $\mathcal{D}$
$\delta_{p_t}\mathcal{H}_{\mathcal{B}}$	:	Variational derivative of $\mathcal{H}_{\mathcal{B}}$ with respect to p_t
$\mathcal{X}_{cl}$	:	Configuration space for the closed-loop system
$\mathcal{H}_{cl}$	:	Hamiltonian for the closed-loop system
H_c	:	Hamiltonian for the controller
$\mathcal{C}$	:	Casimir functions (functionals)
$\mathcal{X}^*$	:	Equilibrium configuration
$\Vert \bullet \Vert$	:	Norm

Chapter 1

Introduction

The *pendulum on a cart* or the control of a rigid beam (or pendulum) pivoted on a moving cart has been exhaustively investigated in the literature ([1],[14],[18]). The objective here is swing-up from any point in a domain and then stabilization of the pendulum about the vertical upright position. This problem involves two issues - a motion planning to reach the upright position and then a stabilizing control law. The motion planing is often achieved by an energy pumping algorithm. Note that the flexibility of the pendulum is not accounted for in this analysis. Here we examine a physically close parallel to this problem - stabilizing a flexible beam fixed to a moving cart. Motion planning (or an open loop trajectory) is no longer an objective; we are just interested in stabilization of the beam under disturbances. The control effort, once again, is the motion of the cart. Investigations into the control of flexible structures have been made in the past ([6],[8],[13]) and so also investigations into other mechanical systems governed by partial differential equations ([5],[7],[24]). Our objective here is two fold: one being to cast this problem in a new framework that provides better mathematical tools to solve this problem and second, with the beginning made here, investigate alternate control laws for this problem - especially distributed control laws.

We cast our problem in the port-controlled Hamiltonian[1] (PCH) framework, proposed and developed by [20]. PCH systems form a more general class of Hamiltonian systems and are characterized by a Dirac structure. Initial work here focused on theory applicable to systems governed by ordinary differential equations (ODEs) and more recently, the theory of distributed parameter systems (PDEs) and systems with

[1]Named after Irish mathematician **Sir William Rowan Hamilton**.

a combination of ODEs and PDEs ([10],[11]) has been proposed. Independently, the theory of Hamiltonian systems and Dirac structures has been developed by [12]. The approach adopted here is on the lines of [9]. The main idea in Hamiltonian systems is that one uses the notion of *energy* or an *energy-like* function (or functional) to achieve desired objectives. In our problem, both the beam and the cart are mechanical energy storing elements - in the forms of potential energy and kinetic energy. These elements exchange energy between themselves and regulating this process helps in achieving the control objective.

Here we present a stabilizing controller for a vertically upright flexible beam fixed on a moving cart. The flexible beam is a distributed parameter system and the moving cart is a finite dimensional system. We model this system in a mixed finite and infinite dimensional port Hamiltonian (m-pH) framework involving PDEs and ODEs. We view the moving cart and the finite dimensional controller (that actuates the moving cart) as an integrated system. This integrated system is interconnected with the infinite dimensional flexible beam according to a power conserving compatibility condition. To obtain a stabilizing controller the total energy of the closed-loop system is shaped in such a way that the minimum of the total energy corresponds to the desired equilibrium of the closed-loop system. Here we adopt the *energy Casimir method* to accomplish this goal. In the energy Casimir method, we look for Casimir functionals which are basically constants of motion, invariant along any trajectory of the system. Casimir functionals are important in obtaining a stabilizing controller because they relate the controller states with the states of the system. The sufficient conditions for the Casimir functionals of this system are obtained in the fourth chapter of this report. From these conditions we obtain the Casimir functionals.

1.1 Organization of this report

This thesis is organized into 7 chapters as follows:

- In chapter 1, the subject of stabilization of a flexible beam fixed to a moving cart is introduced; with general information regarding the research work carried out as a part of this project.

- In chapter 2, the notion of port Hamiltonian system is introduced for both finite and infinite dimensional systems. We have also shown how Hamiltonian equations are obtained from Euler-Lagrange equations of motion.

- Chapter 3 deals with the modeling of the system. We introduce the notion of power conserving Dirac structure and model the flexible beam as a distributed parameter (infinite dimensional) port Hamiltonian system. Then we model the finite dimensional cart and the controller as an integrated system.

- In chapter 4, we introduce the notion of Casimir functionals which are constant along any trajectory of the closed-loop system. Then we derive the sufficient conditions for the Casimirs of the system under consideration and obtain those Casimir functionals.

- In chapter 5, we prove the asymptotic stability of the desired equilibrium configuration which is the vertically upright position of the flexible beam.

- In chapter 6, the finite dimensional controller is extracted from the integrated system consisting of the cart and the controller. An algorithm to simulate the closed-loop behavior of the system is also provided.

- In chapter 7, a brief summary of the work is given. And we have also pointed out some potential fields to explore based on the work done here.

- In appendix A, we present some relevant topics from the field of differential geometry and exterior algebra.

Chapter 2

Port Hamiltonian system

In this chapter we introduce the notion of port Hamiltonian systems, a special class of passive systems. After giving a brief idea about finite dimensional port Hamiltonian systems, the notion of infinite dimensional (also referred as *distributed parameter*) port Hamiltonian systems is introduced in section (**2.3**).

2.1 Development of Hamiltonian equations from Euler-Lagrange equations of motion

In this section we will derive the Hamiltonian equations of motion [21], starting from the Euler-Lagrange equations of motion. The standard Euler-Lagrange equations are as follows:

$$\frac{d}{dt}\left(\frac{\partial L(q,\dot{q})}{\partial \dot{q}}\right) - \frac{\partial L(q,\dot{q})}{\partial q} = \tau \tag{2.1}$$

where $q = (q_1, \ldots, q_n)^T$ is a vector of generalized configuration coordinates for the system with 'n' degrees of freedom and $\tau = (\tau_1, \ldots, \tau_n)$ is the vector of generalized forces acting on the system.

Let $K(q,\dot{q})$ be the kinetic energy of the system and $P(q)$ be the potential energy of the system. Then the Lagrangian $L(q,\dot{q})$ for the system is given by the difference

$$L(q,\dot{q}) = K(q,\dot{q}) - P(q) \tag{2.2}$$

Now, $\frac{\partial L}{\partial \dot{q}}$ and $\frac{\partial L}{\partial q}$ represents the vectors of partial derivatives of $L(q,\dot{q})$ with respect to $\dot{q}$ and q respectively. For most of the systems, the kinetic energy can be represented as

$$K(q,\dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} \tag{2.3}$$

where the $n \times n$ generalized mass matrix $M(q)$ [sometimes known as inertia matrix] is positive definite and symmetrical (i.e., $M(q) = M^T(q) > 0$). Therefore the Lagrangian for this system is as follows:

$$L(q,\dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} - P(q) \tag{2.4}$$

Now the vector of generalized momenta $p = (p_1, \ldots, p_n)^T$ is defined by $p = \frac{\partial L}{\partial \dot{q}}$ [this is known as Legendre transformation]. Therefore from (2.4), the generalized momenta vector for this case is given by

$$p = M(q)\dot{q} \tag{2.5}$$

$$\text{or,} \quad \dot{q} = M^{-1}(q)p \tag{2.6}$$

The Hamiltonian for a system is its total energy. Therefore for this system the Hamiltonian is given by

$$H(q,\dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} + P(q) \tag{2.7}$$

Now substituting the expression for $\dot{q}$, obtained from (2.6), in (2.7), we get

$$\begin{aligned} H(q,p) &= \frac{1}{2}\big(M^{-1}(q)p\big)^T M(q)\big(M^{-1}(q)p\big) + P(q) \\ &= \frac{1}{2}p^T M^{-1}(q)p + P(q) \end{aligned} \tag{2.8}$$

Therefore,

$$\frac{\partial H(q,p)}{\partial p} = M^{-1}(q)p = \dot{q}$$

Now,

$$\frac{\partial H(q,p)}{\partial q} + \frac{\partial L(q,\dot{q})}{\partial q} = 0$$

Therefore,

$$\frac{\partial L(q,\dot{q})}{\partial q} = -\frac{\partial H(q,p)}{\partial q}$$

From (2.1) we get,

$$\begin{aligned}\dot{p} &= \frac{\partial L(q,\dot{q})}{\partial q} + \tau \\ &= -\frac{\partial H(q,p)}{\partial q} + \tau\end{aligned}$$

In this way by choosing the state vector as $(q_1,\ldots,q_n,p_1,\ldots,p_n)^T$ we transform the n second order equations given in (2.1) to $2n$ first order equations,

$$\dot{q} = \frac{\partial H(q,p)}{\partial p} \tag{2.9}$$

$$\dot{p} = -\frac{\partial H(q,p)}{\partial q} + \tau \tag{2.10}$$

The equations (2.9) and (2.10) are known as Hamiltonian equations of motion and $H(q,p)$ is known as the Hamiltonian of the system.

Now,

$$\begin{aligned}\frac{dH(q,p)}{dt} &= \left(\frac{\partial H}{\partial q}\right)^T \dot{q} + \left(\frac{\partial H}{\partial p}\right)^T \dot{p} \\ &= \left(\frac{\partial H}{\partial p}\right)^T \tau \\ &= \dot{q}^T \tau \end{aligned} \tag{2.11}$$

This energy balance is an immediate result from (2.10) and it says that the change in energy of the system is equal to the supplied work (Conservation of Energy).

Now if it assumed that $P(q)$ is bounded from below then $H(q,p)$ will be also bounded from below as $K(q,p) = \frac{1}{2}p^T M^{-1}(q)p \geq 0$ because $M(q) = M^T(q) > 0$. As $P(q)$ is bounded from below $\exists C > -\infty$ such that $P(q) \geq C$. Then according to [19] this system is passive with respect to inputs $u = \tau$ and outputs $y = \dot{q}$, and with $H(q,p) - C \geq 0$ as the storage function.

The Hamiltonian system in (2.10) can be more generally represented by

$$\begin{aligned} \dot{q} &= \frac{\partial H(q,p)}{\partial p} \\ \dot{p} &= -\frac{\partial H(q,p)}{\partial q} + B(q)u \\ y &= B^T(q)\frac{\partial H(q,p)}{\partial p} \end{aligned} \tag{2.12}$$

where $u, y \in \mathbb{R}^m$ and $B(q)$ is the input force matrix. $B(q)u$ denotes the generalized forces resulting from u. The system will be called under-actuated if $m < n$. And if $m = n$ and $B(q)$ is invertible for each and every value of q then the system is called fully actuated.

Example 2.1.1 *Let us consider a spring mass system (Figure 2.1) with spring constant K and mass M.*

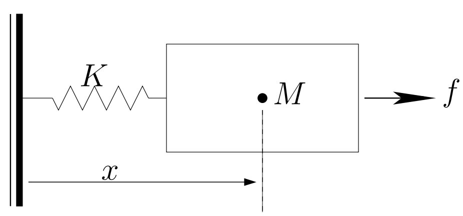

Figure 2.1: A simple mass-spring system

For this System, $q = x$; $p = M\dot{x}$ and the Hamiltonian is: $H = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}Kx^2 = \frac{1}{2M}p^2 + \frac{K}{2}q^2$.

Therefore,

$$\begin{aligned} \frac{\partial H}{\partial q} &= Kq = Kx \\ \text{and,} \qquad \frac{\partial H}{\partial p} &= \frac{p}{M} = \dot{x} = \dot{q} \end{aligned}$$

Now,

$$\begin{aligned} & M\ddot{x} + Kx = f \\ \text{or,} \qquad & \dot{p} = -\frac{\partial H}{\partial q} + f \end{aligned}$$

Therefore, $\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{bmatrix} + \begin{bmatrix} 0 \\ f \end{bmatrix}$ *where* $\tau = f$.
Hence, this system is lossless passive with respect to input $u = f$ *and output* $y = \dot{x}$.

2.2 Finite dimensional port Hamiltonian system

A more generalized class of Hamiltonian systems can be formed by considering a special class of systems which are described in local coordinates by

$$\begin{aligned} \dot{x} &= J(x)\frac{\partial H(x)}{\partial x} + g(x)u \\ y &= g^T(x)\frac{\partial H(x)}{\partial x} \end{aligned} \tag{2.13}$$

where $x = (x_1, \ldots, x_n)$ are local coordinates for a n-dimensional state space manifold $\mathcal{X}$. Also $u, y \in \mathbb{R}^m$. Here $J(x)$ is a skew-symmetric matrix [in a more general sense, skew-adjoint matrix]. Because of the skew-symmetry property of $J(x)$, the energy balance $\frac{d}{dt}(H(x)) = y^T u$ is valid. We call (2.13) a *port Hamiltonian system* with $J(x)$ as a structure matrix and $H(x)$ as Hamiltonian of the system. So, now we are in a position to define a port Hamiltonian system in a more structured fashion.

Definition 2.2.1 ***(port Hamiltonian system)*** *Let* $\mathcal{X}$ *be a* n*-dimensional state space manifold and* $H : \mathcal{X} \to \mathbb{R}$ *be a scalar energy function bounded from below. Let* $J(x)$, $x \in \mathcal{X}$ *be a skew-symmetric matrix. Then the system defined by*

$$\begin{aligned} \dot{x} &= J(x)\frac{\partial H(x)}{\partial x} + g(x)u \\ y &= g^T(x)\frac{\partial H(x)}{\partial x} \end{aligned} \tag{2.14}$$

where $u, y \in \mathbb{R}^m$, *is a port Hamiltonian system with structure matrix* $J(x)$ *and Hamiltonian* $H(x)$.

So we can say in nutshell that a port Hamiltonian system is characterized over a state space manifold $\mathcal{X}$ by a triple $(J(x), g(x), H)$. The pair $(J(x), g(x))$ captures the interconnection structure of the system and $g(x)$ particularly models the port of the system. Another basic property of port Hamiltonian systems is the power balancing property $\frac{d}{dt}(H(x)) = y^T u$ whose physical interpretation is the fact that the internal interconnection structure is power conserving. Port Hamiltonian systems are also *modular* in nature, *i.e.* a power conserving interconnection of a number of port Hamiltonian systems result in another port Hamiltonian system and the new Hamiltonian is just the sum of individual Hamiltonians.

When a dissipative system is modeled as a port Hamiltonian system, the governing equations get a little modification. The governing equations for a port Hamiltonian system with dissipation are

$$\begin{aligned} \dot{x} &= [J(x) - R(x)]\frac{\partial H}{\partial x} + g(x)u \\ y &= g^T(x)\frac{\partial H}{\partial x} \end{aligned} \tag{2.15}$$

where $R(x) = R^T(x) \geq 0$, i.e. $R(x)$ is positive semi-definite and $R(x)$ depends smoothly on x. Also, in this case

$$\begin{aligned} \frac{d}{dt}(H(x)) &= y^T u - \left(\frac{\partial H}{\partial x}\right)^T R(x) \left(\frac{\partial H}{\partial x}\right) \\ &\leq y^T u \end{aligned} \tag{2.16}$$

(2.16) says that the rate of change of internal stored energy can never exceed the power injected to the system from outside.

2.3 Infinite dimensional port Hamiltonian system

Let $\mathcal{D}$ be a n-dimensional spatial domain with $(n-1)$-dimensional boundary $\partial\mathcal{D}$. Now let us define the linear space

$$\mathcal{F}_{p,q} := \Omega^p(\mathcal{D}) \times \Omega^q(\mathcal{D}) \times \Omega^{n-p}(\partial\mathcal{D}) \tag{2.17}$$

where $\Omega^p(\mathcal{D})$ is the space of differential p-forms on the manifold $\mathcal{D}$. And the space $\mathcal{F}_{p,q}$ is named as the space of *flows*. In a similar fashion we also define

$$\mathcal{E}_{p,q} := \Omega^{n-p}(\mathcal{D}) \times \Omega^{n-q}(\mathcal{D}) \times \Omega^{n-q}(\partial\mathcal{D}) \tag{2.18}$$

and this space is named as the space of *efforts*. Therefore the space $\mathcal{F}_{p,q} \times \mathcal{E}_{p,q}$ can be called as the space of port conjugated power variables.

Now, let us consider a space $\mathbb{D} \subset \mathcal{F}_{p,q} \times \mathcal{E}_{p,q}$ which is defined by

$$\mathbb{D} = \left\{ (f_p, f_q, f_b, e_p, e_q, e_b) \in \mathcal{F}_{p,q} \times \mathcal{E}_{p,q} | \begin{bmatrix} f_p \\ f_q \end{bmatrix} = \mathbb{J} \begin{bmatrix} e_p \\ e_q \end{bmatrix}, \begin{bmatrix} f_b \\ e_b \end{bmatrix} = \mathbb{G} \begin{bmatrix} e_p|_{\partial\mathcal{D}} \\ e_q|_{\partial\mathcal{D}} \end{bmatrix} \right\} \tag{2.19}$$

where $\mathbb{J}$ is a skew-adjoint differential operator and $\mathbb{G}$ is a differential operator. We have $\mathbb{D} = \mathbb{D}^{\perp}$. Therefore $\mathbb{D}$ is a Dirac structure[1].

Now consider a Hamiltonian density H

$$H : \Omega^p(\mathcal{D}) \times \Omega^q(\mathcal{D}) \times \mathcal{D} \rightarrow \Omega^n(\mathcal{D}) \tag{2.20}$$

and let's define the total energy as

$$\mathcal{H} = \int_{\mathcal{D}} H \in \mathbb{R} \tag{2.21}$$

Let $\alpha_p, \beta_p \in \Omega^p(\mathcal{D})$, $\alpha_q, \beta_q \in \Omega^q(\mathcal{D})$. Then

$$\begin{aligned} \mathcal{H}(\alpha_p + \epsilon_1\beta_p, \alpha_q + \epsilon_2\beta_q) \; &= \; \int_{\mathcal{D}} H(\alpha_p, \alpha_q, z) + \epsilon_1 \int_{\mathcal{D}} \delta_{\alpha_p}\mathcal{H} \wedge \beta_p \\ &\quad + \epsilon_2 \int_{\mathcal{D}} \delta_{\alpha_q}\mathcal{H} \wedge \beta_q + O({\epsilon_1}^2, {\epsilon_2}^2) \end{aligned}$$

where $\epsilon_1, \epsilon_2 \in \mathbb{R}$, $|\epsilon_1| \ll 1$, $|\epsilon_2| \ll 1$ and $\delta_{\alpha_p}\mathcal{H} \in \Omega^{n-p}(\mathcal{D})$ is the variational derivative[2] of Hamiltonian $\mathcal{H}$ with respect to α_p. Similarly $\delta_{\alpha_q}\mathcal{H} \in \Omega^{n-q}(\mathcal{D})$ is the variational derivative of $\mathcal{H}$ with respect to α_q.

Now consider a time-function

$$(\alpha_p(t), \alpha_q(t)) \in \Omega^p(\mathcal{D}) \times \Omega^q(\mathcal{D})$$

and the Hamiltonian $\mathcal{H}(\alpha_p(t), \alpha_q(t))$ evaluated along this trajectory. Then,

$$\frac{d\mathcal{H}}{dt} = \int_{\mathcal{D}} \left[\delta_{\alpha_p}\mathcal{H} \wedge \frac{\partial \alpha_p}{\partial t} + \delta_{\alpha_q}\mathcal{H} \wedge \frac{\partial \alpha_q}{\partial t} \right] \tag{2.22}$$

at any time t. The differential forms $\frac{\partial \alpha_p}{\partial t}$ and $\frac{\partial \alpha_q}{\partial t}$ represent the generalized velocities

[1]Refer section **(A.2)** for further details on Dirac structure.
[2]Refer section **(A.6)** for further details on variational derivative.

of the energy variables α_p, α_q. They are connected to the Dirac structure $\mathbb{D}$ by setting

$$\begin{array}{rcl} f_p & = & -\frac{\partial \alpha_p}{\partial t} \\ f_q & = & -\frac{\partial \alpha_q}{\partial t} \end{array} \tag{2.23}$$

Here the minus sign is included to have a consistent energy flow description. We also set

$$\begin{array}{rcl} e_p & = & \delta_{\alpha_p}\mathcal{H} \\ e_q & = & \delta_{\alpha_q}\mathcal{H} \end{array} \tag{2.24}$$

The *distributed port Hamiltonian system* with n-dimensional spatial domain $\mathcal{D}$, state-space $\Omega^p(\mathcal{D}) \times \Omega^q(\mathcal{D})$, the Dirac structure $\mathbb{D}$ given by (2.19) and Hamiltonian $\mathcal{H}$, is given as

$$\begin{array}{rcl} \begin{bmatrix} \frac{\partial \alpha_p}{\partial t} \\ \frac{\partial \alpha_q}{\partial t} \end{bmatrix} & = & -\mathbb{J} \begin{bmatrix} \delta_{\alpha_p}\mathcal{H} \\ \delta_{\alpha_q}\mathcal{H} \end{bmatrix} \\ \\ \begin{bmatrix} f_b \\ e_b \end{bmatrix} & = & \mathbb{G} \begin{bmatrix} \delta_{\alpha_p}\mathcal{H}|_{\partial\mathcal{D}} \\ \delta_{\alpha_q}\mathcal{H}|_{\partial\mathcal{D}} \end{bmatrix} \end{array} \tag{2.25}$$

Chapter 3

Modeling the system

In this chapter we define a Dirac structure (a subspace of the space of power variables) and model the flexible beam as an infinite dimensional port Hamiltonian system. During modeling, we consider the cart and the controller as an integrated system because by doing so we can separate the finite and the infinite dimensional part of the overall system. Then we define a power conserving interconnection between the finite and the infinite dimensional part of the system.

3.1 Modeling the beam

The system of our interest is shown in the Figure (3.1). The beam of length L is fixed to the cart at the bottom. The cart is actuated by a horizontal force F. The equilibrium configuration of the beam is vertically upright. The deflection of the beam from the equilibrium configuration is $w(l,t)$ and the rotation of the cross-section of the beam due to bending is $\phi(l,t)$, where $l \in [0,L]$ is the spatial coordinate along the length of the beam in its equilibrium position. The displacement of the cart in the horizontal direction is represented by $x(t)$. We represent the spatial domain by $\mathcal{D} := [0,L]$. The boundary of the spatial domain $\mathcal{D}$ is represented by $\partial\mathcal{D} = \{0,L\}$. Since the lower end of the beam ($l = 0$) is fixed to the cart, $\phi(0,t) = 0 \quad \forall t$.

The coefficients ρ, I_ρ, E and I represent the mass per unit length, the mass moment of inertia of the cross section, Young's modulus and the moment of inertia of the cross-section, respectively. The coefficient K has the dimension of force. The potential energy of the beam is a function of the shear and the bending. The kinetic energy is a function of the translational and rotational momenta. With these parameters and

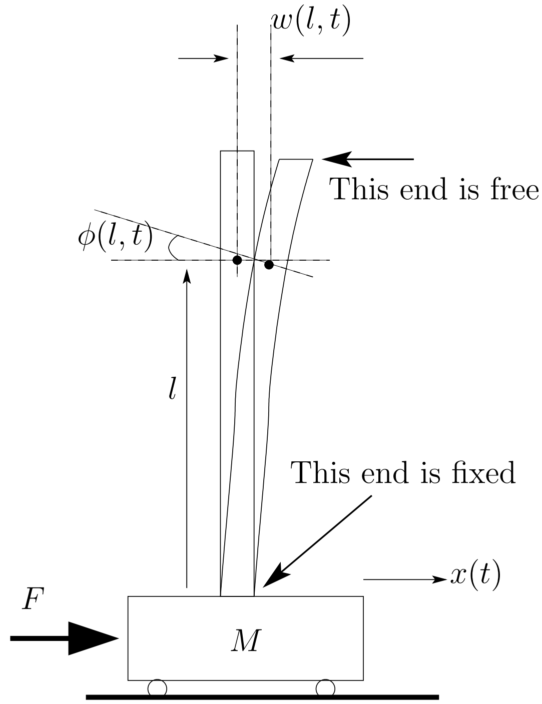

Figure 3.1: A flexible beam on a cart

notation, the total energy in the beam consists of the following components:

- Translational Kinetic Energy: $\frac{1}{2}\int_0^L \rho\left(\frac{\partial w}{\partial t}+\dot{x}\right)^2 dl$
- Rotational Kinetic Energy: $\frac{1}{2}\int_0^L I_\rho\left(\frac{\partial \phi}{\partial t}\right)^2 dl$
- Translational Strain Energy: $\frac{1}{2}\int_0^L K\left(\phi-\frac{\partial w}{\partial l}\right)^2 dl$
- Rotational Strain Energy: $\frac{1}{2}\int_0^L EI\left(\frac{\partial \phi}{\partial l}\right)^2 dl$
- Gravitational Potential Energy: $\frac{\rho L^2 g}{2}$

Note that under the assumptions, the gravitational potential remains a constant. Since the cart traverses a horizontal plane, its total energy is just the kinetic energy given by

$$\frac{1}{2}M\dot{x}^2$$

The Hamiltonian for a mechanical system being the total energy is thus given by

$$\mathcal{H} = \frac{1}{2}M\dot{x}^2+\frac{1}{2}\int_0^L\left[\rho\left(\frac{\partial w}{\partial t}+\dot{x}\right)^2+I_\rho\left(\frac{\partial \phi}{\partial t}\right)^2+K\left(\phi-\frac{\partial w}{\partial l}\right)^2+EI\left(\frac{\partial \phi}{\partial l}\right)^2\right]dl$$

$$+\frac{\rho L^2 g}{2} \tag{3.1}$$

The vertically upright beam is *fixed* to the cart. If the cart is viewed as one system and the beam the second, the interconnection between the beam and the cart must be expressed appropriately in the modeling framework using compatible boundary conditions. Note that the flexible beam is an infinite dimensional system governed by a PDE, and the cart is a finite dimensional port Hamiltonian system. In the PCH framework, the compatibility of the interconnection is achieved through a power conserving condition which implies that no energy is dissipated in the interconnection. We denote the Hamiltonian for the flexible beam as $\mathcal{H}_{\mathcal{B}}$. Denoting

$$\begin{aligned}
z &\triangleq w + x \\
\text{we have} \quad \frac{\partial z}{\partial t} &= \frac{\partial w}{\partial t} + \dot{x} \\
\text{and} \quad \frac{\partial z}{\partial l} &= \frac{\partial w}{\partial l} \qquad (\text{as } \frac{\partial x}{\partial l} 0) \\
z(l,t) &= w(l,t) + x(t) \qquad \forall t \\
z(0,t) &= x(t) \qquad \forall t
\end{aligned} \tag{3.2}$$

With these assumptions, the Hamiltonian for the beam ($\mathcal{H}_{\mathcal{B}}$) can be rewritten as

$$\begin{aligned}
\mathcal{H}_{\mathcal{B}} &= \frac{1}{2}\int_0^L \left[\rho\left(\frac{\partial z}{\partial t}\right)^2 + I_\rho\left(\frac{\partial \phi}{\partial t}\right)^2 + K\left(\phi - \frac{\partial z}{\partial l}\right)^2 + EI\left(\frac{\partial \phi}{\partial l}\right)^2\right] dl \\
&\quad + \frac{\rho L^2 g}{2}
\end{aligned} \tag{3.3}$$

3.1.1 The Dirac structure for the flexible beam

The notion of a Dirac structure [3] is essential to modeling the system in a PCH framework. A Dirac structure is the mathematical interpretation of the notion of energy-conserving systems. Further, to do so we need to employ the framework of differential geometry. Notice that the displacement/velocity variables are functions of two independent variables - the spatial l and the temporal t. In the language of differential geometry, the spatial domain $[0, L]$ is a *manifold with a boundary* which we denote as $\mathcal{D} := [0, L]$ with the notation for the boundary being $\partial\mathcal{D} = \{0, L\}$. This is a manifold of dimension 1.

Our objective now is to rewrite the Hamiltonian in the language of differential geometry employing the concepts of forms, star operators, wedge operators and an integral over a manifold. The notion of *differential forms* extends the common notion of a differential to a manifold with certain additional properties. A form also helps to define an integral over a manifold. Consider the following 1-forms on the 1-dimensional manifold $\mathcal{D}$

$$\begin{array}{ll} \epsilon_t(l,t) \triangleq \left(\frac{\partial z}{\partial l} - \phi\right) dl & \epsilon_r(l,t) \triangleq \left(\frac{\partial \phi}{\partial l}\right) dl \\ p_t(l,t) \triangleq \rho\left(\frac{\partial z}{\partial t}\right) dl & p_r(l,t) \triangleq I_\rho\left(\frac{\partial \phi}{\partial t}\right) dl \end{array} \tag{3.4}$$

where ϵ denotes strain, p denotes momentum and subscripts t and r denote translation and rotation. The *Hodge star operator* on a manifold transforms a differential k-form on a n-dimensional differential manifold into a differential $(n-k)$-form. Here, corresponding to the 1-forms on $\mathcal{D}$ we have 0-forms given by

$$\begin{array}{ll} \star\epsilon_t = \left(\frac{\partial z}{\partial l} - \phi\right) & \star\epsilon_r = \frac{\partial \phi}{\partial l} \\ \star p_t = \rho\left(\frac{\partial z}{\partial t}\right) & \star p_r = I_\rho\left(\frac{\partial \phi}{\partial t}\right) \end{array} \tag{3.5}$$

where $\star$ represents the *Hodge star operator* which transforms a differential k-form on a n-dimensional differential manifold into a differential $(n-k)$-form. Let $\Psi^k(\mathcal{D})$ represent the space of k-forms on $\mathcal{D}$, that is the space of alternating k-linear forms on $\mathcal{D}$. Then $\epsilon_t, \epsilon_r, p_t, p_r \in \Psi^1(\mathcal{D})$ and $z, \phi \in \Psi^0(\mathcal{D})$. A zero-form is just a function.

Using these above defined differential forms, the expression for $\mathcal{H}_\mathcal{B}$ can be written as

$$\mathcal{H}_\mathcal{B} = \frac{1}{2}\int_\mathcal{D}\left[\frac{1}{\rho}(\star p_t)\wedge p_t + \frac{1}{I_\rho}(\star p_r)\wedge p_r + K(\star\epsilon_t)\wedge\ \epsilon_t + EI(\star\epsilon_r)\wedge\epsilon_r\right] + \frac{1}{2}\rho g L^2 \tag{3.6}$$

where $\wedge$ stands for the wedge product between differential forms. The rate of change of the Hamiltonian $\mathcal{H}_\mathcal{B}$ at any time t is given by

$$\begin{aligned} \frac{d\mathcal{H}_\mathcal{B}}{dt} &= \int_\mathcal{D}\left[(\delta_{p_t}\mathcal{H}_\mathcal{B})\wedge\frac{\partial p_t}{\partial t} + (\delta_{p_r}\mathcal{H}_\mathcal{B})\wedge\frac{\partial p_r}{\partial t} + (\delta_{\epsilon_t}\mathcal{H}_\mathcal{B})\wedge\frac{\partial \epsilon_t}{\partial t} + (\delta_{\epsilon_r}\mathcal{H}_\mathcal{B})\wedge\frac{\partial \epsilon_r}{\partial t}\right] \\ &= \int_\mathcal{D}\left[(\frac{1}{\rho}\star p_t)\wedge\frac{\partial p_t}{\partial t} + (\frac{1}{I_\rho}\star p_r)\wedge\frac{\partial p_r}{\partial t} + (K\star\epsilon_t)\wedge\frac{\partial \epsilon_t}{\partial t} + (EI\star\epsilon_r)\wedge\frac{\partial \epsilon_r}{\partial t}\right] \end{aligned}$$

where $\delta_{p_t}\mathcal{H}_\mathcal{B}, \delta_{p_r}\mathcal{H}_\mathcal{B}, \delta_{\epsilon_t}\mathcal{H}_\mathcal{B}$ and $\delta_{\epsilon_r}\mathcal{H}_\mathcal{B}$ represent the variational derivatives of the

Hamiltonian $\mathcal{H}_{\mathcal{B}}$ and are known as *efforts* and they represent generalized forces. The differential forms $\frac{\partial p_t}{\partial t}, \frac{\partial p_r}{\partial t}, \frac{\partial \epsilon_t}{\partial t}$ and $\frac{\partial \epsilon_r}{\partial t}$ denote the time derivatives of the the 1-forms p_t, p_r, ϵ_t and ϵ_r are known as *flows* and they represent the generalized velocities. These flows and efforts define what is termed as a power variable - the product of a flow (generalized velocity) and effort (generalized force) gives a generalized power unit. For a finite-dimensional system with no input (like a spring-mass or extensions of the same), conservative means that the rate of change of the total energy (H_f) is zero or

$$\frac{dH_f}{dt} = 0$$

The notion of boundary conditions does not exist for such a system. However, for an infinite-dimensional system, the notion of a boundary is important and for the beam

$$\frac{d\mathcal{H}_{\mathcal{B}}}{dt} = \quad \text{Power dissipated at the boundary} \tag{3.7}$$

This relationship now helps us to formulate a distributed port Hamiltonian model for the beam by defining a Dirac structure for the model. To get the Dirac structure, we need to have a well defined space of power variables.

The space of flows $\mathcal{F}$ is now defined as

$$(f_{p_t}, f_{p_r}, f_{\epsilon_t}, f_{\epsilon_r}, f_b^t, f_b^r) = (-\frac{\partial p_t}{\partial t}, -\frac{\partial p_r}{\partial t}, -\frac{\partial \epsilon_t}{\partial t}, -\frac{\partial \epsilon_r}{\partial t}, (f_b^t(0), f_b^t(L)), (f_b^r(0), f_b^r(L))) \in \mathcal{F}$$

where

$$\mathcal{F} := \underbrace{\Psi^1(\mathcal{D}) \times \Psi^1(\mathcal{D}) \times \Psi^1(\mathcal{D}) \times \Psi^1(\mathcal{D})}_{\text{Flows (Generalized velocities)}} \times \underbrace{\Psi^0(\partial\mathcal{D}) \times \Psi^0(\partial\mathcal{D})}_{\text{Flow at border}} \tag{3.8}$$

Notice the last two terms in the 6-tuple with a subscript b denote the flows at the boundary due to translation and rotation. A physical interpretation to these will be given shortly.

Theorem 3.1.1 *Now we state a standard result from differential geometry [15]. Let $\mathcal{Q}$ be a n-dimensional differentiable manifold. If $\alpha \in \Psi^k(\mathcal{Q})$ and $\beta \in \Psi^{n-k}(\mathcal{Q})$, then their duality product is defined as $\langle \beta | \alpha \rangle : \int_{\mathcal{Q}} \alpha \wedge \beta$. This is also valid for the boundary $\partial\mathcal{Q}$.*

The space of effort $\mathcal{E}$ thus is the dual of $\mathcal{F}$.

$$(e_{p_t}, e_{p_r}, e_{\epsilon_t}, e_{\epsilon_r}, e_b^t, e_b^r) = (\delta_{p_t}\mathcal{H}_\mathcal{B}, \delta_{p_r}\mathcal{H}_\mathcal{B}, \delta_{\epsilon_t}\mathcal{H}_\mathcal{B}, \delta_{\epsilon_r}\mathcal{H}_\mathcal{B}, (e_b^t(0), e_b^t(L)), (e_b^r(0), e_b^r(L))) \in \mathcal{E}$$

where

$$\mathcal{E} := \underbrace{\Psi^0(\mathcal{D}) \times \Psi^0(\mathcal{D}) \times \Psi^0(\mathcal{D}) \times \Psi^0(\mathcal{D})}_{\text{Efforts (Generalized forces)}} \times \underbrace{\Psi^0(\partial\mathcal{D}) \times \Psi^0(\partial\mathcal{D})}_{\text{Effort at border}} \tag{3.9}$$

The power dissipated at the boundary of the beam is

$$f_b(L) \wedge e_b(L) - f_b(0) \wedge e_b(0)$$

The duality product is defined as

$$\begin{aligned}&\langle (e_{p_t}, e_{p_r}, e_{\epsilon_t}, e_{\epsilon_r}, e_b^t, e_b^r), (f_{p_t}, f_{p_r}, f_{\epsilon_t}, f_{\epsilon_r}, f_b^t, f_b^r) \rangle \\ &\quad := \int_\mathcal{D} [f_{p_t} \wedge e_{p_t} + f_{p_r} \wedge e_{p_r} + f_{\epsilon_t} \wedge e_{\epsilon_t} + f_{\epsilon_r} \wedge e_{\epsilon_r}] + \int_{\partial\mathcal{D}} [f_b^t \wedge e_b^t + f_b^r \wedge e_b^r]\end{aligned}$$

Using the definition of duality product we define a bilinear form over $\mathcal{F} \times \mathcal{E}$ as

$$\begin{aligned}&\ll (f_{p_t}^1, \cdot, f_b^{r,1}, e_{p_t}^1, \cdot, e_b^{r,1}), (f_{p_t}^2, \cdot, f_b^{r,2}, e_{p_t}^2, \cdot, e_b^{r,2}) \gg \\ &:= \int_\mathcal{D} [f_{p_t}^1 \wedge e_{p_t}^2 + f_{p_r}^1 \wedge e_{p_r}^2 + f_{\epsilon_t}^1 \wedge e_{\epsilon_t}^2 + f_{\epsilon_r}^1 \wedge e_{\epsilon_r}^2] \\ &\quad + \int_\mathcal{D} [f_{p_t}^2 \wedge e_{p_t}^1 + f_{p_r}^2 \wedge e_{p_r}^1 + f_{\epsilon_t}^2 \wedge e_{\epsilon_t}^1 + f_{\epsilon_r}^2 \wedge e_{\epsilon_r}^1] \\ &\quad + \int_{\partial\mathcal{D}} [f_b^{t,1} \wedge e_b^{t,2} + f_b^{r,1} \wedge e_b^{r,2} + f_b^{t,2} \wedge e_b^{t,1} + f_b^{r,2} \wedge e_b^{r,1}]\end{aligned} \tag{3.10}$$

where $(f_{p_t}^i, \ldots, f_b^{r,i}, e_{p_t}^i, \ldots, e_b^{r,i}) \in \mathcal{F} \times \mathcal{E}$ for $i = 1, 2$. Using this definition, the Dirac structure is proposed.

Proposition 3.1.2 *Let us consider the space of power variables $\mathcal{F} \times \mathcal{E}$ with $\mathcal{F}$ and $\mathcal{E}$ defined by (3.8) and (3.9) respectively and the bilinear form (3.10). Define a subspace $\mathbb{D} \subset \mathcal{F} \times \mathcal{E}$ as follows*

$$\mathbb{D} \;=\; \Big\{ (f_{p_t}, f_{p_r}, f_{\epsilon_t}, f_{\epsilon_r}, f_b^t, f_b^r, e_{p_t}, e_{p_r}, e_{\epsilon_t}, e_{\epsilon_r}, e_b^t, e_b^r) \in \mathcal{F} \times \mathcal{E} |$$

$$\left.\begin{bmatrix} f_{p_t} \\ f_{p_r} \\ f_{\epsilon_t} \\ f_{\epsilon_r} \end{bmatrix} = -\begin{bmatrix} 0 & 0 & d & 0 \\ 0 & 0 & \star & d \\ d & -\star & 0 & 0 \\ 0 & d & 0 & 0 \end{bmatrix}\begin{bmatrix} e_{p_t} \\ e_{p_r} \\ e_{\epsilon_t} \\ e_{\epsilon_r} \end{bmatrix} ; \begin{bmatrix} f_b^t \\ f_b^r \\ e_b^t \\ e_b^r \end{bmatrix} = \begin{bmatrix} e_{p_t}|_{\partial\mathcal{D}} \\ e_{p_r}|_{\partial\mathcal{D}} \\ e_{\epsilon_t}|_{\partial\mathcal{D}} \\ e_{\epsilon_r}|_{\partial\mathcal{D}} \end{bmatrix}\right\} \quad (3.11)$$

where $|_{\partial\mathcal{D}}$ represents the restriction on the boundary of the spatial domain $\mathcal{D}$. Then $\mathbb{D}$ satisfies the property $\mathbb{D} = \mathbb{D}^{\perp}$ and $\mathbb{D}$ is a Dirac structure.

Proof To prove $\mathbb{D} = \mathbb{D}^{\perp}$, at first we show $\mathbb{D} \subseteq \mathbb{D}^{\perp}$ and then show $\mathbb{D}^{\perp} \subseteq \mathbb{D}$.

Let us assume,

$$w_i = (f_{p_t}^i, \ldots, f_b^{r,i}, e_{p_t}^i, \ldots, e_b^{r,i}) \in \mathcal{F} \times \mathcal{E}$$

with $i = 1, 2$. If $\forall w_1, w_2 \in \mathbb{D}$, $\ll w_1, w_2 \gg = 0$, then $w_1, w_2 \in \mathbb{D}^{\perp}$. This way we can show $\mathbb{D} \subseteq \mathbb{D}^{\perp}$.

$$\begin{aligned}
\ll w_1, w_2 \gg &= \int_{\mathcal{D}} \big[-de_{\epsilon_t}^1 \wedge e_{p_t}^2 - \star e_{\epsilon_t}^1 \wedge e_{p_r}^2 - de_{\epsilon_r}^1 \wedge e_{p_r}^2 - de_{p_t}^1 \wedge e_{\epsilon_t}^2 \\
&\quad + \star\, e_{p_r}^1 \wedge e_{\epsilon_t}^2 - de_{p_r}^1 \wedge e_{\epsilon_r}^2\big] + \int_{\mathcal{D}} \big[-de_{\epsilon_t}^1 \wedge e_{p_t}^2 - \star e_{\epsilon_t}^1 \wedge e_{p_r}^2 \\
&\quad - de_{\epsilon_r}^1 \wedge e_{p_r}^2 - de_{p_t}^1 \wedge e_{\epsilon_t}^2 + \star e_{p_r}^1 \wedge e_{\epsilon_t}^2 - de_{p_r}^1 \wedge e_{\epsilon_r}^2\big] \\
&\quad + \int_{\partial\mathcal{D}} \big[f_b^{t,1} \wedge e_b^{t,2} + f_b^{r,1} \wedge e_b^{r,2} + f_b^{t,2} \wedge e_b^{t,1} + f_b^{r,2} \wedge e_b^{r,1}\big] \\
&= -\int_{\mathcal{D}} d\big[e_{p_t}^1 \wedge e_{\epsilon_t}^2 + e_{p_r}^1 \wedge e_{\epsilon_r}^2 + e_{p_t}^2 \wedge e_{\epsilon_t}^1 + e_{p_r}^2 \wedge e_{\epsilon_r}^1\big] \\
&\qquad + \int_{\partial\mathcal{D}} \big[f_b^{t,1} \wedge e_b^{t,2} + f_b^{r,1} \wedge e_b^{r,2} + f_b^{t,2} \wedge e_b^{t,1} + f_b^{r,2} \wedge e_b^{r,1}\big] \\
&\qquad\qquad \Big(\text{Since, } d(e^1 \wedge e^2) = de^1 \wedge e^2 + e^1 \wedge de^2 \text{ as } e^1, e^2 \in \Psi^0(\mathcal{D}).\Big) \\
&= -\int_{\partial\mathcal{D}} \big[e_{p_t}^1 \wedge e_{\epsilon_t}^2 + e_{p_r}^1 \wedge e_{\epsilon_r}^2 + e_{p_t}^2 \wedge e_{\epsilon_t}^1 + e_{p_r}^2 \wedge e_{\epsilon_r}^1\big] \\
&\qquad + \int_{\partial\mathcal{D}} \big[f_b^{t,1} \wedge e_b^{t,2} + f_b^{r,1} \wedge e_b^{r,2} + f_b^{t,2} \wedge e_b^{t,1} + f_b^{r,2} \wedge e_b^{r,1}\big] \\
&\qquad\qquad \Big(\text{Using Stokes' theorem.}\Big) \\
&= 0
\end{aligned}$$

Hence, $\mathbb{D} \subseteq \mathbb{D}^{\perp}$.

To show $\mathbb{D}^{\perp} \subseteq \mathbb{D}$, let us assume $w_2 \in \mathbb{D}^{\perp}$. Then $\forall w_1 \in \mathbb{D}$, we have $\ll w_1, w_2 \gg = 0$.

$$\therefore \int_{\mathcal{D}} \left[-de^1_{\epsilon_t} \wedge e^2_{p_t} - \star e^1_{\epsilon_t} \wedge e^2_{p_r} - de^1_{\epsilon_r} \wedge e^2_{p_r} - de^1_{p_t} \wedge e^2_{\epsilon_t} + \star e^1_{p_r} \wedge e^2_{\epsilon_t} - de^1_{p_r} \wedge e^2_{\epsilon_r}\right]$$
$$\int_{\mathcal{D}} \left[f^2_{p_t} \wedge e^1_{p_t} + f^2_{p_r} \wedge e^1_{p_r} + f^2_{\epsilon_t} \wedge e^1_{\epsilon_t} + f^2_{\epsilon_r} \wedge e^1_{\epsilon_r}\right] + \int_{\partial\mathcal{D}} \left[e^1_{p_t}|_{\partial\mathcal{D}} \wedge e^{t,2}_b\right.$$
$$\left.+ e^1_{p_r}|_{\partial\mathcal{D}} \wedge e^{r,2}_b + f^{t,2}_b \wedge e^1_{\epsilon_t}|_{\partial\mathcal{D}} + f^{r,2}_b \wedge e^1_{\epsilon_r}|_{\partial\mathcal{D}}\right] = 0$$

$$\Rightarrow \int_{\mathcal{D}} \left[e^1_{\epsilon_t} \wedge de^2_{p_t} - d(e^1_{\epsilon_t} \wedge e^2_{p_t}) + e^1_{\epsilon_r} \wedge de^2_{p_r} - d(e^1_{\epsilon_r} \wedge e^2_{p_r}) + e^1_{p_t} \wedge de^2_{\epsilon_t} - d(e^1_{p_t} \wedge e^2_{\epsilon_t})\right.$$
$$\left.+ e^1_{p_r} \wedge de^2_{\epsilon_r} - d(e^1_{p_r} \wedge e^2_{\epsilon_r}) + e^1_{p_r} \wedge \star e^2_{\epsilon_t} - e^1_{\epsilon_t} \wedge \star e^2_{p_r}\right] + \int_{\mathcal{D}} \left[e^1_{p_t} \wedge f^2_{p_t} + e^1_{p_r} \wedge f^2_{p_r}\right.$$
$$\left.+ e^1_{\epsilon_t} \wedge f^2_{\epsilon_t} + e^1_{\epsilon_r} \wedge f^2_{\epsilon_r}\right] + \int_{\partial\mathcal{D}} \left[e^1_{p_t}|_{\partial\mathcal{D}} \wedge e^{t,2}_b + e^1_{p_r}|_{\partial\mathcal{D}} \wedge e^{r,2}_b + f^{t,2}_b \wedge e^1_{\epsilon_t}|_{\partial\mathcal{D}}\right.$$
$$\left.+ f^{r,2}_b \wedge e^1_{\epsilon_r}|_{\partial\mathcal{D}}\right] = 0$$

(Using the properties of exterior derivative and wedge product)

$$\Rightarrow \int_{\mathcal{D}} \left[e^1_{\epsilon_t} \wedge (de^2_{p_t} - \star e^2_{p_r} + f^2_{\epsilon_t}) + e^1_{\epsilon_r} \wedge (de^2_{p_r} + f^2_{e_r}) + e^1_{p_t} \wedge (de^2_{\epsilon_t} + f^2_{p_t})\right.$$
$$\left.+ e^1_{p_r} \wedge (de^2_{\epsilon_r} + \star e^2_{\epsilon_t} + f^2_{p_r})\right] - \int_{\partial\mathcal{D}} \left[e^1_{\epsilon_t} \wedge e^2_{p_t} + e^1_{\epsilon_r} \wedge e^2_{p_r} + e^1_{p_t} \wedge e^2_{\epsilon_t} + e^1_{p_r} \wedge e^2_{\epsilon_r}\right]$$
$$\int_{\partial\mathcal{D}} \left[e^1_{p_t}|_{\partial\mathcal{D}} \wedge e^{t,2}_b + e^1_{p_r}|_{\partial\mathcal{D}} \wedge e^{r,2}_b + f^{t,2}_b \wedge e^1_{\epsilon_t}|_{\partial\mathcal{D}} + f^{r,2}_b \wedge e^1_{\epsilon_r}|_{\partial\mathcal{D}}\right] = 0$$

(Rearranging and using Stokes' theorem)

Therefore,

$$\begin{aligned} f^2_{p_t} &= -de^2_{\epsilon_t} \\ f^2_{p_r} &= -\star e^2_{\epsilon_t} - de^2_{\epsilon_r} \\ f^2_{\epsilon_t} &= -de^2_{p_t} + \star e^2_{p_r} \\ f^2_{\epsilon_r} &= -de^2_{p_r} \end{aligned}$$

Also the restrictions on the boundary $\partial\mathcal{D}$ are satisfied. So, $w_2 \in \mathbb{D}$.

Therefore, $\mathbb{D}^\perp \subseteq \mathbb{D}$.

Hence, $\mathbb{D} = \mathbb{D}^\perp$. ∎

In simpler words, this subspace $\mathbb{D}$ defines the permissible space for our system flows and efforts, since it satisfies equation (3.7). Therefore the distributed port Hamiltonian model for the flexible beam is given by

$$\frac{\partial}{\partial t}\begin{bmatrix} p_t \\ p_r \\ \epsilon_t \\ \epsilon_r \end{bmatrix} = \begin{bmatrix} 0 & 0 & d & 0 \\ 0 & 0 & \star & d \\ d & -\star & 0 & 0 \\ 0 & d & 0 & 0 \end{bmatrix}\begin{bmatrix} \delta_{p_t}\mathcal{H}_{\mathcal{B}} \\ \delta_{p_r}\mathcal{H}_{\mathcal{B}} \\ \delta_{\epsilon_t}\mathcal{H}_{\mathcal{B}} \\ \delta_{\epsilon_r}\mathcal{H}_{\mathcal{B}} \end{bmatrix};$$

$$\begin{bmatrix} f_b^t \\ f_b^r \\ e_b^t \\ e_b^r \end{bmatrix} = \begin{bmatrix} \delta_{p_t}\mathcal{H}_{\mathcal{B}}|_{\partial\mathcal{D}} \\ \delta_{p_r}\mathcal{H}_{\mathcal{B}}|_{\partial\mathcal{D}} \\ \delta_{\epsilon_t}\mathcal{H}_{\mathcal{B}}|_{\partial\mathcal{D}} \\ \delta_{\epsilon_r}\mathcal{H}_{\mathcal{B}}|_{\partial\mathcal{D}} \end{bmatrix} \tag{3.12}$$

with respect to the Dirac structure $\mathbb{D}$ (3.11).

3.2 Modeling the cart and the controller

Recall that we consider the controller and the cart as an integrated system. The motivation behind doing so is to separate the finite and infinite dimensional parts of our overall system. The cart is a typical second order system governed by one configuration variable (the displacement x) and we look for a controller with one configuration variable ($\tilde{x}$). Denoting the combined Hamiltonian function of the cart and the beam as H_c, the port Hamiltonian formulation for this integrated system is assumed to be

$$\begin{bmatrix} \dot{q}_c \\ \dot{p}_c \end{bmatrix} = \left(\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & D_c \end{bmatrix}\right)\begin{bmatrix} \frac{\partial H_c}{\partial q_c} \\ \frac{\partial H_c}{\partial p_c} \end{bmatrix} + \begin{bmatrix} 0 \\ G_c \end{bmatrix} f_c \tag{3.13}$$

$$\text{and,} \quad e_c = G_c^T \partial_{p_c} H_c \tag{3.14}$$

where $q_c = [q_{c_1}, q_{c_2}]^T = [x, \tilde{x}]^T \in Q_C \subset \mathbb{R}^2$ are the generalized co-ordinates for the integrated system and f_c, e_c are the power conjugated port variables. Since the only component of actuation is the horizontal force on the cart, we assume $f_{c2} = 0$; The components of The energy dissipation in the integrated system is taken into account by the matrix $D_c = D_c^T \geq 0$. From this point onwards, we will refer to this finite dimensional integrated system as the *controller*.

3.3 Interconnection between the finite dimensional and the infinite dimensional system

As the finite dimensional system and the infinite dimensional flexible beam are interconnected according to a power conserving compatibility condition, the net energy going out from the controller $-e_c^T f_c$ is equal to the net energy injected to the flexible beam $e_b(L) \wedge f_b(L) - e_b(0) \wedge f_b(0)$. This yields

$$f_c^T e_c = f_b(0) \wedge e_b(0) - f_b(L) \wedge e_b(L) \tag{3.15}$$

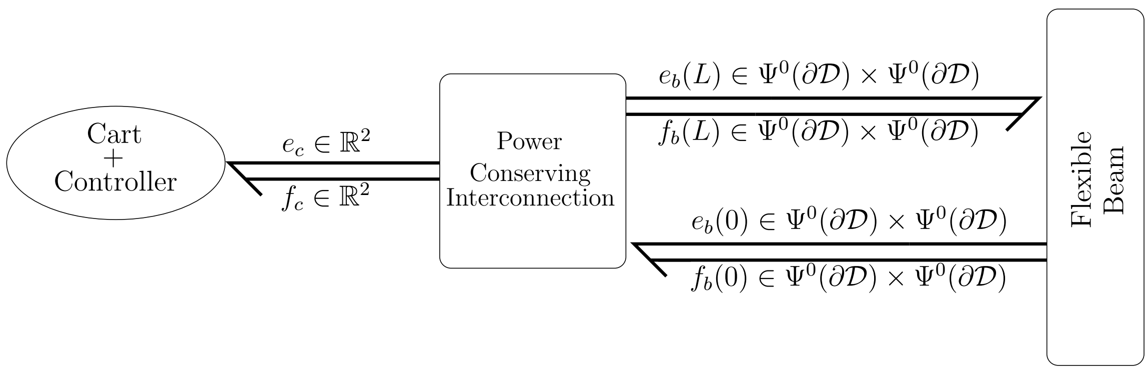

Figure 3.2: Bond-graph representation of the closed-loop system

From the physics of the problem, we have

$$\begin{aligned}
f_b^t(0) &= \delta_{p_t}\mathcal{H}_\mathcal{B}(0) = \frac{\partial z}{\partial t}(0,t) = \text{Velocity of the cart} \\
f_b^r(0) &= \delta_{p_r}\mathcal{H}_\mathcal{B}(0) = \frac{\partial \phi}{\partial t}(0,t) = 0 = \text{Angular velocity of the beam cross-section} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{at the base} \\
e_b^t(0) &= \delta_{\epsilon_t}\mathcal{H}_\mathcal{B}(0) = K(\frac{\partial z}{\partial l} - \phi)(0,t) \\
e_b^r(0) &= \delta_{\epsilon_r}\mathcal{H}_\mathcal{B}(0) = EI\frac{\partial \phi}{\partial l}(0,t)
\end{aligned}$$

The closed-loop system is a mixed finite and infinite dimensional port Hamiltonian (m-pH) system. The Hamiltonian for the closed-loop system ($\mathcal{H}_{cl}$) is defined over the extended configuration space $\mathcal{X}_{cl}$

$$\mathcal{X}_{cl} := \underbrace{T^*Q_c}_{\mathcal{X}_c} \times \underbrace{\Psi^1(\mathcal{D}) \times \Psi^1(\mathcal{D}) \times \Psi^1(\mathcal{D}) \times \Psi^1(\mathcal{D})}_{\mathcal{X}_\infty} \tag{3.16}$$

and it is given by the sum of the individual Hamiltonians as

$$\mathcal{H}_{cl} := \mathcal{H}_{\mathcal{B}} + H_c \tag{3.17}$$

where $\mathcal{X}_c$ and $\mathcal{X}_\infty$ are respectively the configuration spaces for the controller and the flexible beam.

Our objective in control design is to obtain a controller which will drive the system towards the desired equilibrium, in this case the upright position of the beam and a stationary cart. With this in mind, we adopt the energy- Casimir approach. We *shape* the total energy $\mathcal{H}_{cl}$ by making a proper choice for the controller Hamiltonian (H_c) in order to have a minimum of $\mathcal{H}_{cl}$ at the desired equilibrium of the closed-loop system. And to shape the total energy $\mathcal{H}_{cl}$, at first we will look for *Casimir functionals* for the closed-loop system.

Chapter 4

Casimir functionals

This chapter introduces the notion of Casimir functionals which are constants of motion, invariant of the Hamiltonian chosen for the controller. The Casimir also gives the relation between the controller states and the states of the system. Here we assume a special form for the Casimirs to obtain a direct expression for the controller dynamics. After obtaining the sufficient conditions for the Casimir, the Casimirs for the closed-loop system are also obtained.

4.1 Energy Casimir method

Casimirs on a Poisson manifold $\mathcal{P}$ are invariants along all flows on $\mathcal{P}$. They denote a notion stronger than just constants of motion that correspond to a particular flow. Thus if $C : \mathcal{P} \rightarrow \mathbb{R}$ is a Casimir, then in the notation of [12]

$$\{C, F\} = 0 \ \forall F \in \mathcal{F}(\mathcal{P})$$

where $\{\cdot, \cdot\}$ denotes the Poisson bracket of two functions. So if one obtains a Casimir for the Poisson manifold defined by the state space ($\mathcal{X}_{cl}$) of the autonomous (controller + beam) system, then since the controller is as yet undetermined, one could choose appropriate dynamics for the controller by an appropriate choice of the Hamiltonian which incorporates the additional constraint of stability.

Definition 4.1.1 ***(Casimir functionals)*** *Consider a scalar function $\mathcal{C} : \mathcal{X}_{cl} \rightarrow \mathbb{R}$ defined on the extended configuration space (3.16). Then $\mathcal{C}$ is a Casimir functional* *for the m-pH if and only if*

$$\frac{d\mathcal{C}}{dt} = 0 \quad \forall \mathcal{H}_{cl} : \mathcal{X}_{cl} \to \mathbb{R} \tag{4.1}$$

where $\mathcal{H}_{cl} = \mathcal{H}_B + H_c$. *(Recall* $H_c = H_{cart} + H_{controller}$*)*

Writing $\mathcal{H}_{cl} = \mathcal{H}_B + H_c$, the existence of Casimir functionals $\mathcal{C}_1, \cdots, \mathcal{C}_k$ also ensures

$$\frac{d}{dt}\left(\mathcal{H}_{cl} + f(\mathcal{C}_1, \cdots, \mathcal{C}_k)\right) = 0 \tag{4.2}$$

for any function $f : \mathbb{R}^k \to \mathbb{R}$. Hence, even if $\mathcal{H}_{cl}$ is not positive definite at an equilibrium $x_{eq} \in \mathcal{X}_{cl}$, then $\mathcal{H}_{cl} + f(\mathcal{C}_1, \cdots, \mathcal{C}_k)$ may be positive definite at x_{eq} by a proper choice of f, and in this way it may serve as a *Lyapunov function.* This method for stability analysis is known as the *Energy Casimir method* [12].

Now consider a Casimir for the closed-loop system composed of the beam, the cart and the controller

$$\mathcal{C}(q_c, p_c, p_t, p_r, \epsilon_t, \epsilon_r) = \quad \text{constant}$$

The Casimir establishes a relationship between the controller states and the states of the beam and hence this technique is referred to as control by interconnection and energy shaping. Alternatively, it can be said that the closed-loop trajectory is constrained to evolve on a particular sub-manifold of the extended configuration space given in (3.16). In particular, here, we consider the Casimir functionals of the form

$$\mathcal{C}_i(q_c, p_c, p_t, p_r, \epsilon_t, \epsilon_r) := q_{c_i} + \tilde{\mathcal{C}}_i(p_c, p_t, p_r, \epsilon_t, \epsilon_r) \tag{4.3}$$

where $i = 1, 2$. Differentiating the above leads to a direct expression for the controller dynamics as

$$\dot{q}_{c_i} = -\frac{d\tilde{\mathcal{C}}_i(p_c, p_t, p_r, \epsilon_t, \epsilon_r)}{dt}$$

4.2 Conditions for existence of a Casimir

We now derive sufficient conditions for the existence of a Casimir. As the Casimir is defined over the configuration space $\mathcal{X}_c \times \mathcal{X}_\infty$, we can write

$$\begin{aligned} \frac{d\mathcal{C}}{dt} &= \left(\frac{\partial \mathcal{C}}{\partial p_c}\right)^T \dot{p}_c + \left(\frac{\partial \mathcal{C}}{\partial q_c}\right)^T \dot{q}_c \\ &\quad + \int_{\mathcal{D}} \left[\delta_{p_t}\mathcal{C} \wedge \frac{\partial p_t}{\partial t} + \delta_{p_r}\mathcal{C} \wedge \frac{\partial p_r}{\partial t} + \delta_{\epsilon_t}\mathcal{C} \wedge \frac{\partial \epsilon_t}{\partial t} + \delta_{\epsilon_r}\mathcal{C} \wedge \frac{\partial \epsilon_r}{\partial t}\right] \end{aligned} \tag{4.4}$$

Now comparing the equations (3.12), (3.13) and (4.4), we get

$$\begin{aligned} \frac{d\mathcal{C}}{dt} &= \left(\frac{\partial \mathcal{C}}{\partial p_c}\right)^T \left(-\frac{\partial H_c}{\partial q_c} - D_c \frac{\partial H_c}{\partial p_c} + G_c f_c\right) + \left(\frac{\partial \mathcal{C}}{\partial q_c}\right)^T \left(\frac{\partial H_c}{\partial p_c}\right) \\ &\quad + \int_{\mathcal{D}} \Big[\delta_{p_t}\mathcal{C} \wedge d(\delta_{\epsilon_t}\mathcal{H_B}) + \delta_{p_r}\mathcal{C} \wedge \Big(* \, \delta_{\epsilon_t}\mathcal{H_B} + d(\delta_{\epsilon_r}\mathcal{H_B})\Big) \\ &\qquad + \delta_{\epsilon_t}\mathcal{C} \wedge \Big(d(\delta_{p_t}\mathcal{H_B}) - *\delta_{p_r}\mathcal{H_B}\Big) + \delta_{\epsilon_r}\mathcal{C} \wedge d(\delta_{p_r}\mathcal{H_B})\Big] \end{aligned}$$

Using the *properties of the exterior derivative* and employing *Stokes' Theorem*, we get the simplified form of the equation (4.4) as the following:

$$\begin{aligned} \frac{d\mathcal{C}}{dt} &= -\left(\frac{\partial \mathcal{C}}{\partial p_c}\right)^T \frac{\partial H_c}{\partial q_c} + \left[{G_c}^T \frac{\partial \mathcal{C}}{\partial p_c}\right] f_c + \left[\left(\frac{\partial \mathcal{C}}{\partial q_c}\right)^T - \left(\frac{\partial \mathcal{C}}{\partial p_c}\right)^T D_c\right] \frac{\partial H_c}{\partial p_c} \\ &\quad + \begin{bmatrix} \delta_{\epsilon_t}\mathcal{H_B} \\ \delta_{\epsilon_r}\mathcal{H_B} \end{bmatrix}^T \begin{bmatrix} \delta_{p_t}\mathcal{C} \\ \delta_{p_r}\mathcal{C} \end{bmatrix} \Bigg|_{\partial\mathcal{D}=0}^{\partial\mathcal{D}L} + \begin{bmatrix} \delta_{p_t}\mathcal{H_B} \\ \delta_{p_r}\mathcal{H_B} \end{bmatrix}^T \begin{bmatrix} \delta_{\epsilon_t}\mathcal{C} \\ \delta_{\epsilon_r}\mathcal{C} \end{bmatrix} \Bigg|_{\partial\mathcal{D}=0}^{\partial\mathcal{D}L} \\ &\quad + \int_{\mathcal{D}} \Big[\delta_{\epsilon_t}\mathcal{H_B} \wedge \Big(d(\delta_{p_t}\mathcal{C}) - *\delta_{p_r}\mathcal{C}\Big) + \delta_{\epsilon_r}\mathcal{H_B} \wedge d(\delta_{p_r}\mathcal{C}) \\ &\quad + \delta_{p_t}\mathcal{H_B} \wedge d(\delta_{\epsilon_t}\mathcal{C}) + \delta_{p_r}\mathcal{H_B} \wedge \Big(d(\delta_{\epsilon_r}\mathcal{C}) + *\delta_{\epsilon_t}\mathcal{C}\Big)\Big] \end{aligned} \tag{4.5}$$

Now, as the Casimir is constant along any trajectory of the closed-loop system, the integral in the equation (4.5) should be zero and hence,

$$d(\delta_{p_t}\mathcal{C}) - *\delta_{p_r}\mathcal{C} = 0 \tag{4.6}$$

$$d(\delta_{p_r}\mathcal{C}) = 0 \tag{4.7}$$

$$d(\delta_{\epsilon_t}\mathcal{C}) = 0 \tag{4.8}$$

$$d(\delta_{\epsilon_r}\mathcal{C}) + *\delta_{\epsilon_t}\mathcal{C} = 0 \tag{4.9}$$

Similarly,

$$\frac{\partial \mathcal{C}}{\partial p_c} = 0 \tag{4.10}$$

The above five conditions reduce the equation (4.5) to yield the sixth condition:

$$\begin{aligned}\left(\frac{\partial \mathcal{C}}{\partial q_c}\right)^T \frac{\partial H_c}{\partial p_c} &+ \delta_{p_t}\mathcal{C}|_L \wedge \delta_{\epsilon_t}\mathcal{H}_{\mathcal{B}}|_L - \delta_{p_t}\mathcal{C}|_0 \wedge \delta_{\epsilon_t}\mathcal{H}_{\mathcal{B}}|_0 + \delta_{p_r}\mathcal{C}|_L \wedge \delta_{\epsilon_r}\mathcal{H}_{\mathcal{B}}|_L \\ &-\delta_{p_r}\mathcal{C}|_0 \wedge \delta_{\epsilon_r}\mathcal{H}_{\mathcal{B}}|_0 + \delta_{\epsilon_t}\mathcal{C}|_L \wedge \delta_{p_t}\mathcal{H}_{\mathcal{B}}|_L - \delta_{\epsilon_t}\mathcal{C}|_0 \wedge \delta_{p_t}\mathcal{H}_{\mathcal{B}}|_0 \\ &+\delta_{\epsilon_r}\mathcal{C}|_L \wedge \delta_{p_r}\mathcal{H}_{\mathcal{B}}|_L - \delta_{\epsilon_r}\mathcal{C}|_0 \wedge \delta_{p_r}\mathcal{H}_{\mathcal{B}}|_0 = 0 \end{aligned} \tag{4.11}$$

These six conditions (4.6), (4.7), (4.8), (4.9), (4.10) and (4.11) provide sufficient conditions for the Casimir functionals.

4.3 Casimir functionals for the system under investigation

The conditions derived in the previous section will be used in this section to obtain Casimir functionals for the closed-loop system. Recall that we look for Casimirs of the form

$$\mathcal{C}_i(q_c, p_c, p_t, p_r, \epsilon_t, \epsilon_r) := q_{c_i} + \tilde{\mathcal{C}}_i(p_c, p_t, p_r, \epsilon_t, \epsilon_r) \tag{4.12}$$

with $i = 1, 2$.

Equation (4.10) straightaway tells us that $\mathcal{C}_i$, hence $\tilde{\mathcal{C}}_i$, is independent of p_c. Integrating (4.7) and (4.8), we get

$$\delta_{p_r}\mathcal{C}_i = constant = k_1^i \tag{4.13}$$
$$\delta_{\epsilon_t}\mathcal{C}_i = constant = k_2^i \tag{4.14}$$

From (4.6),

$$\begin{aligned} d(\delta_{p_t}\mathcal{C}_i) &= *\delta_{p_r}\mathcal{C}_i = *k_1^i k_1^i dl \\ \Rightarrow \delta_{p_t}\mathcal{C}_i &= k_3^i + k_1^i l \end{aligned} \tag{4.15}$$

where the constants k_1^i and k_3^i are given by $k_1^i = \delta_{p_r}\mathcal{C}_i|_{\partial\mathcal{D}=0}$ and $k_3^i = \delta_{p_t}\mathcal{C}_i|_{\partial\mathcal{D}0}$.Similarly from (4.9),

$$\delta_{\epsilon_r}\mathcal{C}_i = k_4^i - k_2^i l \tag{4.16}$$

where the constants k_2^i and k_4^i are given by $k_2^i = \delta_{\epsilon_t}\mathcal{C}_i|_{\partial\mathcal{D}0}$ and $k_4^i = \delta_{\epsilon_r}\mathcal{C}_i|_{\partial\mathcal{D}=0}$.

Using the results obtained in (4.13), (4.14), (4.15) and (4.16), the equation (4.11) can be rewritten as:

$$\begin{aligned}\left(\frac{\partial\mathcal{C}_i}{\partial q_c}\right)^T\left(\frac{\partial H_c}{\partial p_c}\right) &+ k_1^i(\delta_{\epsilon_r}\mathcal{H}_\mathcal{B}|_L - \delta_{\epsilon_r}\mathcal{H}_\mathcal{B}|_0) + k_2^i(\delta_{p_t}\mathcal{H}_\mathcal{B}|_L - \delta_{p_t}\mathcal{H}_\mathcal{B}|_0)\\ &+k_3^i(\delta_{\epsilon_t}\mathcal{H}_\mathcal{B}|_L - \delta_{\epsilon_t}\mathcal{H}_\mathcal{B}|_0) + k_4^i(\delta_{p_r}\mathcal{H}_\mathcal{B}|_L - \delta_{p_r}\mathcal{H}_\mathcal{B}|_0) + k_1^i L\delta_{\epsilon_t}\mathcal{H}_\mathcal{B}|_L\\ &-k_2^i L\delta_{p_r}\mathcal{H}_\mathcal{B}|_L = 0\end{aligned} \tag{4.17}$$

From the assumed form of the Casimir in equation (4.12), we obtain

$$\left(\frac{\partial\mathcal{C}}{\partial q_c}\right) = \begin{bmatrix} \frac{\partial\mathcal{C}_1}{\partial q_{c_1}} & \frac{\partial\mathcal{C}_1}{\partial q_{c_2}} \\ \frac{\partial\mathcal{C}_2}{\partial q_{c_1}} & \frac{\partial\mathcal{C}_2}{\partial q_{c_2}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{4.18}$$

Hence, from (4.13), (4.14), (4.15), (4.16) and (4.18) we conclude that

$$\mathcal{C}_i = q_{c_i} + \int_\mathcal{D}\left[(k_3^i + k_1^i l)p_t + k_1^i p_r + k_2^i\epsilon_t + (k_4^i - k_2^i l)\epsilon_r\right] \tag{4.19}$$

(where $i \in \{1, 2\}$) are the Casimir functionals for the closed-loop system. For every energy function H_c of the combined system, we have

$$q_{c_i} = -\int_\mathcal{D}\left[(k_3^i + k_1^i l)p_t + k_1^i p_r + k_2^i\epsilon_t + (k_4^i - k_2^i l)\epsilon_r\right] + \mathcal{C}_i \tag{4.20}$$

as $\mathcal{C}_i$'s are invariant along any trajectory of the system ($\mathcal{C}_1$ and $\mathcal{C}_2$ depend on the initial condition). Since H_c can be any arbitrary function over T^*Q_c, the total energy of the closed-loop system can be given a desired shape to achieve a minimum of energy at a desired equilibrium.

At this point, we choose the controller port variable e_c as the following:

$$\begin{aligned} e_c &= \begin{bmatrix} e_{c_1} \\ e_{c_2} \end{bmatrix} \\ &= \begin{bmatrix} k_2^1 \\ k_2^2 \end{bmatrix}\delta_{p_t}\mathcal{H}_\mathcal{B}|_0 - \begin{bmatrix} k_2^1 \\ k_2^2 \end{bmatrix}\delta_{p_t}\mathcal{H}_\mathcal{B}|_L + \begin{bmatrix} k_4^1 \\ k_4^2 \end{bmatrix}\delta_{p_r}\mathcal{H}_\mathcal{B}|_0 - \begin{bmatrix} k_4^1 - k_2^1 L \\ k_4^2 - k_2^2 L \end{bmatrix}\delta_{p_r}\mathcal{H}_\mathcal{B}|_L\end{aligned}$$

$$+\begin{bmatrix} k_3^1 \\ k_3^2 \end{bmatrix} \delta_{\epsilon_t}\mathcal{H}_{\mathcal{B}}|_0 - \begin{bmatrix} k_3^1 + k_1^1 L \\ k_3^2 + k_1^2 L \end{bmatrix} \delta_{\epsilon_t}\mathcal{H}_{\mathcal{B}}|_L + \begin{bmatrix} k_1^1 \\ k_1^2 \end{bmatrix} \delta_{\epsilon_r}\mathcal{H}_{\mathcal{B}}|_0 - \begin{bmatrix} k_1^1 \\ k_1^2 \end{bmatrix} \delta_{\epsilon_r}\mathcal{H}_{\mathcal{B}}|_L \tag{4.21}$$

With this particular choice for e_c, we obtain the following relation from (4.17)

$$\begin{aligned} \left(\frac{\partial \mathcal{C}}{\partial q_c}\right)^T \left(\frac{\partial H_c}{\partial p_c}\right) - e_c &= 0 \\ \Rightarrow \qquad \left(\left(\frac{\partial \mathcal{C}}{\partial q_c}\right)^T - G_c^T\right)\left(\frac{\partial H_c}{\partial p_c}\right) &= 0 \end{aligned}$$

Hence, we have

$$G_c = \left(\frac{\partial \mathcal{C}}{\partial q_c}\right) \tag{4.22}$$

Chapter 5

Stability analysis

In this chapter we define the desired equilibrium configuration of the closed-loop system which is composed of the flexible beam (infinite dimensional part) and the cart & the controller (finite dimensional part). Then we prove the stability of this equilibrium configuration by showing that the equilibrium is a strict extremum of the total energy of the closed-loop system ($\mathcal{H}_{cl}$).

5.1 Equilibrium configuration

The desired equilibrium configuration of the flexible beam is the vertically upright position. Now, it can be shown that when the flexible beam is vertically upright, we have

$$\left.\begin{array}{rcl} p_t^* &=& 0 \\ p_r^* &=& 0 \\ \epsilon_t^* &=& 0 \\ \epsilon_r^* &=& 0 \end{array}\right\} \tag{5.1}$$

From (4.20) and (5.1), we can define the equilibrium values of q_{c_1} and q_{c_2} as:

$$\begin{aligned} q_{c_1}^* &= q_{c_1}(p_t^*, p_r^*, \epsilon_t^*, \epsilon_r^*) \\ &= \mathcal{C}_1 \end{aligned} \tag{5.2}$$

$$\begin{aligned} q_{c_2}^* &= q_{c_2}(p_t^*, p_r^*, \epsilon_t^*, \epsilon_r^*) \\ &= \mathcal{C}_2 \end{aligned} \tag{5.3}$$

Also, at the equilibrium

$$p_c^* = 0 \tag{5.4}$$

The Hamiltonian of the finite dimensional controller will be chosen in order to regulate the closed-loop system in the restricted sub-manifold

$$\mathcal{X}^* = (p_t^*, p_r^*, \epsilon_t^*, \epsilon_r^*, p_c^*)$$

We choose the finite dimensional controller Hamiltonian (H_c) as:

$$H_c = \frac{1}{2}{p_c}^T {M_c}^{-1} p_c + \frac{1}{2}K_{c1}(q_{c_1} - q_{c_1}^*)^2 + \frac{1}{2}K_{c2}(q_{c_2} - q_{c_2}^*)^2 + \psi(q_{c_1}, q_{c_2}) \tag{5.5}$$

where $M_c = {M_c}^T > 0$, $K_{c1} > 0$, $K_{c2} > 0$ and the function ψ will be defined in the later part of this section. In the remaining part of this section we will prove that by the above choice of H_c (5.5), the configuration $\mathcal{X}^*$ is stable.

5.2 Stability analysis

We will now prove the stability of the closed-loop system by showing that the equilibrium is a strict extremum of the total energy of the closed-loop system. The definition of stability in the sense of Lyapunov for mixed finite and infinite dimensional system can be given as follows [9].

Definition 5.2.1 ***(Lyapunov stability for mixed systems)*** *The equilibrium configuration $\mathcal{X}^*$ for a mixed finite and infinite dimensional system is said to be stable in the sense of Lyapunov with respect to the norm $||\cdot||$ if, for every $\epsilon > 0$, there exists $\delta_\epsilon > 0$ such that*

$$||\mathcal{X}(0) - \mathcal{X}^*|| < \delta_\epsilon \Rightarrow ||\mathcal{X}(t) - \mathcal{X}^*|| < \epsilon$$

for every $t > 0$, where $\mathcal{X}(0)$ is the initial configuration of the mixed system.

In order to verify the stability of the equilibrium configuration ($\mathcal{X}^*$), the following conditions must hold (as proposed in [16])

- The equilibrium configuration ($\mathcal{X}^*$) should be an extremum of the closed-loop Hamiltonian ($\mathcal{H}_{cl}$); that is,

$$\nabla\mathcal{H}_{cl}(\mathcal{X}^*) = 0 \tag{5.6}$$

- Let us introduce the nonlinear functional

$$\mathcal{N}(\Delta\mathcal{X}) \triangleq \mathcal{H}_{cl}(\mathcal{X}^* + \Delta\mathcal{X}) - \mathcal{H}_{cl}(\mathcal{X}^*) \tag{5.7}$$

where $\Delta\mathcal{X}$ is the displacement from the equilibrium configuration $\mathcal{X}^*$. The configuration $\mathcal{X}^*$ will be stable if there exists some $\gamma_1, \gamma_2, \alpha > 0$ such that

$$\gamma_1 \|\Delta\mathcal{X}\|^2 \leq \mathcal{N}(\Delta\mathcal{X}) \leq \gamma_2 \|\Delta\mathcal{X}\|^\alpha \tag{5.8}$$

This is known as the *convexity condition* [16].

Let's represent the state variable of the closed-loop system by $\mathcal{X}$. From (3.6), (3.17) and (5.5), we can write the Hamiltonian for the closed-loop system as

$$\begin{aligned}\mathcal{H}_{cl} &= \mathcal{H}_{\mathcal{B}} + H_c \\ &= \frac{1}{2}\int_{\mathcal{D}} \left[\frac{1}{\rho}(\star p_t) \wedge p_t + \frac{1}{I_\rho}(\star p_r) \wedge p_r + K(\star\epsilon_t) \wedge \ \epsilon_t + EI(\star\epsilon_r) \wedge \epsilon_r\right] + \frac{1}{2}\rho g L^2 \\ &\quad + \frac{1}{2}{p_c}^T {M_c}^{-1} p_c + \frac{1}{2}K_{c1}(q_{c_1} - q_{c_1}^*)^2 + \frac{1}{2}K_{c2}(q_{c_2} - q_{c_2}^*)^2 + \psi(q_{c_1}, q_{c_2}) \end{aligned} \tag{5.9}$$

The first step in the stability analysis is to find out the suitable function $\psi(q_{c_1}, q_{c_2})$ such that (5.6) is satisfied. Now,

$$\nabla\mathcal{H}_{cl}(\mathcal{X}) = \begin{bmatrix} \delta_{p_t}\mathcal{H}_{cl} \\ \delta_{p_r}\mathcal{H}_{cl} \\ \delta_{\epsilon_t}\mathcal{H}_{cl} \\ \delta_{\epsilon_r}\mathcal{H}_{cl} \\ \partial_{p_c}\mathcal{H}_{cl} \end{bmatrix}$$

Clearly,

$$\partial_{p_c}\mathcal{H}_{cl}(\mathcal{X}^*) = {M_c}^{-1} p_c^* = 0$$

Also,

$$\begin{aligned}\delta_{p_t}\mathcal{H}_{cl}(\mathcal{X}) &= \frac{1}{\rho}(\star p_t) - K_{c1}(q_{c_1} - q_{c_1}^*)(k_3^1 + k_1^1 l) - K_{c2}(q_{c_2} - q_{c_2}^*)(k_3^2 + k_1^2 l) \\ &\quad - \frac{\partial\psi}{\partial q_{c_1}}(k_3^1 + k_1^1 l) - \frac{\partial\psi}{\partial q_{c_2}}(k_3^2 + k_1^2 l)\end{aligned}$$

$$\Rightarrow \quad \delta_{p_t}\mathcal{H}_{cl}(\mathcal{X}^*) \;=\; -\frac{\partial\psi}{\partial q_{c_1}}(k_3^1 + k_1^1 l) - \frac{\partial\psi}{\partial q_{c_2}}(k_3^2 + k_1^2 l)$$

In a similar fashion, we obtain

$$\begin{aligned}
\delta_{p_r}\mathcal{H}_{cl}(\mathcal{X}^*) &= -\frac{\partial\psi}{\partial q_{c_1}}k_1^1 - \frac{\partial\psi}{\partial q_{c_2}}k_1^2 \\
\delta_{\epsilon_t}\mathcal{H}_{cl}(\mathcal{X}^*) &= -\frac{\partial\psi}{\partial q_{c_1}}k_2^1 - \frac{\partial\psi}{\partial q_{c_2}}k_2^2 \\
\delta_{\epsilon_r}\mathcal{H}_{cl}(\mathcal{X}^*) &= -\frac{\partial\psi}{\partial q_{c_1}}(k_4^1 - k_2^1 l) - \frac{\partial\psi}{\partial q_{c_2}}(k_4^2 - k_2^2 l)
\end{aligned}$$

As the condition $\nabla\mathcal{H}_{cl}(\mathcal{X}^*) = 0$ must be satisfied for $\mathcal{X}^*$ to be stable, we have the following restrictions upon ψ

$$\begin{bmatrix} k_3^1 + k_1^1 l & k_3^2 + k_1^2 l \\ k_1^1 & k_1^2 \\ k_2^1 & k_2^2 \\ k_4^1 - k_2^1 l & k_4^2 - k_2^2 l \end{bmatrix} \begin{bmatrix} \frac{\partial\psi}{\partial q_{c_1}} \\ \frac{\partial\psi}{\partial q_{c_2}} \end{bmatrix} = 0$$

Therefore

$$\frac{\partial\psi}{\partial q_{c_1}} = \frac{\partial\psi}{\partial q_{c_2}} = 0$$

Hence, we can assume

$$\psi(q_{c_1}, q_{c_2}) = \Psi = Constant \tag{5.10}$$

Therefore, the closed-loop Hamiltonian ($\mathcal{H}_{cl}$) can be written as

$$\begin{aligned}
\mathcal{H}_{cl} &= \frac{1}{2}\int_{\mathcal{D}}\Big[\frac{1}{\rho}(\star p_t)\wedge p_t + \frac{1}{I_\rho}(\star p_r)\wedge p_r + K(\star\epsilon_t)\wedge\ \epsilon_t + EI(\star\epsilon_r)\wedge\epsilon_r\Big] \\
&\quad + \frac{1}{2}{p_c}^T {M_c}^{-1} p_c + \frac{1}{2}K_{c1}(q_{c_1} - q_{c_1}^*)^2 + \frac{1}{2}K_{c2}(q_{c_2} - q_{c_2}^*)^2 + \tilde{\Psi}
\end{aligned} \tag{5.11}$$

where $\tilde{\Psi} = \Psi + \frac{1}{2}\rho g L^2 = Constant.$

Now we have to verify the convexity condition (5.8) and after calculation we get the nonlinear functional $\mathcal{N}(\Delta\mathcal{X})$ as

$$\begin{aligned}
\mathcal{N}(\Delta\mathcal{X}) &= \mathcal{H}_{cl}(\mathcal{X}^* + \Delta\mathcal{X}) - \mathcal{H}_{cl}(\mathcal{X}^*) \\
&= \frac{1}{2}\int_{\mathcal{D}}\Big[\frac{1}{\rho}(\star\Delta p_t)\wedge\Delta p_t + \frac{1}{I_\rho}(\star\Delta p_r)\wedge\Delta p_r + K(\star\Delta\epsilon_t)\wedge\ \Delta\epsilon_t
\end{aligned}$$

$$
\begin{aligned}
&+ EI(\star\Delta\epsilon_r)\wedge\Delta\epsilon_r\Big] \\
&+\frac{1}{2}K_{c1}\Big(\int_{\mathcal{D}}\left[(k_3^1+k_1^1 l)\Delta p_t + k_1^1\Delta p_r + k_2^1\Delta\epsilon_t + (k_4^1-k_2^1 l)\Delta\epsilon_r\right]\Big)^2 \\
&+\frac{1}{2}K_{c2}\Big(\int_{\mathcal{D}}\left[(k_3^2+k_1^2 l)\Delta p_t + k_1^2\Delta p_r + k_2^2\Delta\epsilon_t + (k_4^2-k_2^2 l)\Delta\epsilon_r\right]\Big)^2 \\
&+\frac{1}{2}\Delta {p_c}^T {M_c}^{-1}\Delta p_c \qquad (5.12)
\end{aligned}
$$

We define the norm as

$$
\begin{aligned}
\|\Delta\mathcal{X}\|^2 \;\triangleq\; &\int_{\mathcal{D}}\left[(\star\Delta p_t)\wedge\Delta p_t + (\star\Delta p_r)\wedge\Delta p_r + (\star\Delta\epsilon_t)\wedge\ \Delta\epsilon_t + (\star\Delta\epsilon_r)\wedge\Delta\epsilon_r\right] \\
&+\Delta {p_c}^T\Delta p_c \qquad (5.13)
\end{aligned}
$$

Now, (5.8) can be satisfied by choosing γ_1, γ_2 and α as

$$
\begin{aligned}
\gamma_1 &= \frac{1}{2}\min\left\{\frac{1}{\rho},\frac{1}{I_\rho},K,EI,\ \min\left\{\mathrm{eig}({M_c}^{-1})\right\}\right\} && (5.14)\\
\alpha &= 2 && (5.15)\\
\gamma_2 &= \tilde{\gamma_2}\cdot\max\Big\{8L\Big[(k_3^1)^2+(k_1^1)^2L+(k_3^2)^2+(k_1^2)^2L\Big]+1, \\
&\qquad 4L\Big[(k_1^1)^2+(k_1^2)^2\Big]+1, 4L\Big[(k_2^1)^2+(k_2^2)^2\Big]+1, \\
&\qquad 8L\Big[(k_4^1)^2+(k_2^1)^2L+(k_4^2)^2+(k_2^2)^2L\Big]+1\Big\} && (5.16)
\end{aligned}
$$

where

$$
\tilde{\gamma_2} = \frac{1}{2}\max\Big\{\frac{1}{\rho},\frac{1}{I_\rho},K,EI,\ \max\left\{\mathrm{eig}({M_c}^{-1})\right\},K_{c1},K_{c2}\Big\} \qquad (5.17)
$$

We also have

$$
\begin{aligned}
\frac{d\mathcal{H}_{cl}}{dt} &= -\Big(\frac{\partial H_c}{\partial p_c}\Big)^T D_c\Big(\frac{\partial H_c}{\partial p_c}\Big) \\
&\leq 0 \qquad (5.18)
\end{aligned}
$$

and $\frac{d\mathcal{H}_{cl}}{dt}(\mathcal{X}^*)=0$. In this way the asymptotic stability of the equilibrium configuration is proved.

Chapter 6

Finite dimensional controller

In this chapter we extract the finite dimensional controller from the integrated system. The stabilization of the flexible beam is achieved by attaching the cart to a spring and applying a horizontal force to it. We made a couple of attempts to simulate the closed-loop behavior of the system, but these attempts have so far not proved fruitful. For simulation purpose, the partial differential equations governing the dynamics of the flexible beam were discretized in space domain using the finite difference method.

6.1 Controller structure and implementation

Without loss of generality the Hamiltonian for the integrated system can be assumed as

$$H_c = \frac{1}{2}{p_c}^T {M_c}^{-1} p_c + \frac{1}{2}K_{c1}(q_{c_1} - \mathcal{C}_1)^2 + \frac{1}{2}K_{c2}(q_{c_2} - \mathcal{C}_2)^2 \tag{6.1}$$

Let's represent the controller state by $\tilde{x}$. And we make the following assumption

$$q_c = \begin{bmatrix} x \\ \tilde{x} \end{bmatrix} \tag{6.2}$$

Also,

$$p_c = M_c \dot{q}_c = \begin{bmatrix} M & 0 \\ 0 & \tilde{M} \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{\tilde{x}} \end{bmatrix} \tag{6.3}$$

With the above assumptions, H_{cart} and $H_{controller}$ can be written as

$$H_{cart} = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}K_{c1}(x - \mathcal{C}_1)^2 \tag{6.4}$$

$$H_{controller} = \frac{1}{2}\tilde{M}\dot{\tilde{x}}^2 + \frac{1}{2}K_{c2}(\tilde{x} - \mathcal{C}_2)^2 \quad (6.5)$$

Now, from (3.13) the following relation can be concluded

$$\begin{bmatrix} \dot{x} \\ \dot{\tilde{x}} \\ M\ddot{x} \\ \tilde{M}\ddot{\tilde{x}} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & -a_1 & -a_2 \\ 0 & -1 & -a_2 & -a_3 \end{bmatrix} \begin{bmatrix} K_{c1}(x - \mathcal{C}_1) \\ K_{c2}(\tilde{x} - \mathcal{C}_2) \\ \dot{x} \\ \dot{\tilde{x}} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} f_{c1} \\ 0 \end{bmatrix} \quad (6.6)$$

where we have assumed $D_c = \begin{bmatrix} a_1 & a_2 \\ a_2 & a_3 \end{bmatrix}$. Also, we have made the assumption "$f_{c_2} = 0$" as no torque is applied at the fixed end of the flexible beam.

From (6.6) we get

$$M\ddot{x} = -K_{c1}(x - \mathcal{C}_1) - a_1\dot{x} - a_2\dot{\tilde{x}} + f_{c1} \quad (6.7)$$

$$\tilde{M}\ddot{\tilde{x}} = -K_{c2}(\tilde{x} - \mathcal{C}_2) - a_2\dot{x} - a_3\dot{\tilde{x}} \quad (6.8)$$

The controller dynamics is governed by (6.8). Initially we assumed that a horizontal force F is applied to the moving cart. Therefore the cart satisfies the following relation

$$M\ddot{x} = -K_{c1}(x - \mathcal{C}_1) + F \quad (6.9)$$

Comparing (6.7) and (6.9), we have

$$F = f_{c1} - a_1\dot{x} - a_2\dot{\tilde{x}} \quad (6.10)$$

It can be concluded that the stabilization is achieved by attaching the cart to a spring of spring constant K_{c1} and applying a horizontal force F to the cart. Another important observation is that the final position of the cart is predefined by the initial configuration of the flexible beam and the initial position of the cart as the final position of the cart depends on the Casimirs and the Casimirs are dependent on the initial parameters of the system. We can also observe that damping is introduced to the cart by application of the horizontal force F thereby avoiding the need for any external damper.

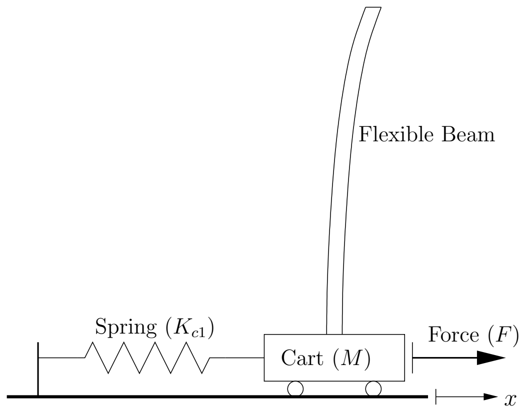

Figure 6.1: Controller implementation

6.2 Simulation

To verify the obtained results we performed a simulation of the closed-loop system and to do that we assumed the following parameters:

- All k_i^j's where $i = 1, 2, 3, 4$ and $j = 1, 2$.
- The positive numbers K_{c1} and K_{c2}.
- A positive definite matrix D_c (i.e., $D_c = D_c^T > 0$).

We can assume that $q_c = \begin{bmatrix} x \\ \tilde{x} \end{bmatrix}$. Then, we have $p_c = M_c \dot{q}_c$. Hence $\dot{x}$ can be computed from $M_c^{-1} p_c$.

For simulation of the closed-loop system one should follow the following algorithm:

S01 Assume the initial configuration profile for the flexible beam, i.e. assume $z(l, 0)$, $\phi(l, 0)$, $\dot{z}(l, 0)$ and $\dot{\phi}(l, 0)$. Also assume $q_c(0)$.

S02 Compute the 1-forms $p_t(l, 0)$, $p_r(l, 0)$, $\epsilon_t(l, 0)$ and $\epsilon_r(l, 0)$ from $z(l, 0)$, $\phi(l, 0)$, $\dot{z}(l, 0)$ and $\dot{\phi}(l, 0)$.

S03 Compute the value of $\mathcal{C}_1$ and $\mathcal{C}_2$ using

$$\mathcal{C}_i = q_{c_i}(0) + \int_{\mathcal{D}} \left[(k_3^i + k_1^i l) p_t(l, 0) + k_1^i p_r(l, 0) + k_2^i \epsilon_t(l, 0) + (k_4^i - k_2^i l) \epsilon_r(l, 0) \right]$$

where $i \in \{1, 2\}$.

S04 We have $\delta_{p_t}\mathcal{H}_{\mathcal{B}}|_0 = \dot{x}$. We also have $\delta_{p_r}\mathcal{H}_{\mathcal{B}}|_0 = 0$ as $\phi(0,t) = 0 \quad \forall t$. Also $\delta_{\epsilon_t}\mathcal{H}_{\mathcal{B}}|_L = 0$ and $\delta_{\epsilon_r}\mathcal{H}_{\mathcal{B}}|_L = 0$ as a consequence of the fact that neither shear force nor bending moment is applied at the free end [23]. So $\delta_{\epsilon_t}\mathcal{H}_{\mathcal{B}}|_0$, $\delta_{\epsilon_r}\mathcal{H}_{\mathcal{B}}|_0$, $\delta_{p_t}\mathcal{H}_{\mathcal{B}}|_L$ and $\delta_{p_r}\mathcal{H}_{\mathcal{B}}|_L$ at $t = 0$ are computed using the initial profile.

S05 Compute e_c at $t = 0$ using the relation

$$\begin{aligned} e_c = &\begin{bmatrix} k_2^1 \\ k_2^2 \end{bmatrix} \dot{x} - \begin{bmatrix} k_2^1 \\ k_2^2 \end{bmatrix} \delta_{p_t}\mathcal{H}_{\mathcal{B}}|_L + \begin{bmatrix} k_3^1 \\ k_3^2 \end{bmatrix} \delta_{\epsilon_t}\mathcal{H}_{\mathcal{B}}|_0 \\ &- \begin{bmatrix} k_4^1 - k_2^1 L \\ k_4^2 - k_2^2 L \end{bmatrix} \delta_{p_r}\mathcal{H}_{\mathcal{B}}|_L + \begin{bmatrix} k_1^1 \\ k_1^2 \end{bmatrix} \delta_{\epsilon_r}\mathcal{H}_{\mathcal{B}}|_0 \end{aligned} \tag{6.11}$$

S06 Using this value of e_c compute

$$p_c = M_c \left[G_c^T\right]^{-1} e_c$$

S07 Compute f_c using the relation

$$f_c^T e_c = f_b(0) \wedge e_b(0) - f_b(L) \wedge e_b(L)$$

Hence f_{c_1} can be easily calculated.

S08 Solve the following equation to update q_c and p_c

$$\begin{bmatrix} \dot{q}_c \\ \dot{p}_c \end{bmatrix} = \begin{bmatrix} 0 & I \\ -I & -D_c \end{bmatrix} \begin{bmatrix} K_{c1}(q_{c_1} - \mathcal{C}_1) \\ K_{c2}(q_{c_1} - \mathcal{C}_1) \\ M_c^{-1} p_c \end{bmatrix} + \begin{bmatrix} 0 \\ G_c \end{bmatrix} f_c$$

S09 Solve (3.12) with the boundary conditions

1. $\delta_{p_t}\mathcal{H}_{\mathcal{B}}|_0 = \dot{x}$
2. $\delta_{p_r}\mathcal{H}_{\mathcal{B}}|_0 = 0$
3. $\delta_{\epsilon_t}\mathcal{H}_{\mathcal{B}}|_L = 0$
4. $\delta_{\epsilon_r}\mathcal{H}_{\mathcal{B}}|_L = 0$

S10 From the updated values of p_t, p_r, ϵ_t and ϵ_r, update z and ϕ. Also update $\delta_{p_t}\mathcal{H}_{\mathcal{B}}|_L$, $\delta_{p_r}\mathcal{H}_{\mathcal{B}}|_L$, $\delta_{\epsilon_t}\mathcal{H}_{\mathcal{B}}|_0$ and $\delta_{\epsilon_r}\mathcal{H}_{\mathcal{B}}|_0$.

S11 Update e_c using (6.11).

S12 Repeat from *S07* to *S12*.

Chapter 7

Conclusion

The main focus of this work has been the stabilization of a vertically upright flexible beam fixed on a moving cart and the control being achieved by an appropriate motion of the cart in the horizontal plane. We modeled the flexible beam as an infinite dimensional port Hamiltonian system. During analysis the cart and the finite dimensional controller are treated as a single integrated system. The correspondence between the states of the integrated system and that of the flexible beam are given by Casimir functionals which are conserved quantities along any trajectory of the system. The power conjugate port variables were also defined in order to satisfy the energy conservation principle. Another important feature of this control methodology is that it is *solution free*; that is the solution of the PDEs defining the system is not required to obtain a stabilizing controller. We also obtained an explicit expression for the combined system Hamiltonian H_c and proved the stability of the closed-loop equilibrium configuration. We have also developed an algorithm to simulate the closed-loop behavior of the system.

7.1 Scope of future work

Although the results presented in this thesis nicely apply the recently developed theory of infinite dimensional port Hamiltonian system to obtain a stabilizing controller for a vertically upright flexible beam, there are quite a few points where further improvement and elaboration is possible. We enumerate these points below.

- In our analysis, we have considered a special form for the Casimir functionals to obtain a direct expression for the controller dynamics. However, this approach yields only a special class of Casimirs. Further study will be required to obtain a more general class of Casimir functionals.

- In the work here, we have only addressed the problem of stabilization of the flexible beam with a fixed end at the cart. Apart from this, one could examine other boundary conditions, varying cross-sections and also further variants of the control aspect - like tracking and optimal control.

- Our attempts at simulation have so far not proved effective. So further efforts can be put out to perform a simulation of the closed-loop system to demonstrate the desired results.

Appendix A

Mathematical preliminaries

As modeling of systems in a distributed port Hamiltonian framework requires knowledge of differential geometry and exterior algebra, here we present some relevant topics from [4], [12], [15] and [17]. Interested readers may consult these references for more details.

A.1 Smooth manifold

A **manifold** $\mathcal{M}$ is a topological space which is *locally* Euclidean (i.e., $\forall x \in \mathcal{M}$, there is some neighborhood U of x and some integer $n \geq 0$ such that U is homeomorphic to $\mathbb{R}^n$). From definition of manifold, it is obvious that any set homeomorphic to a manifold is also a manifold. A **chart** on $\mathcal{M}$ is a subset U of manifold $\mathcal{M}$ together with a homeomorphism $\phi : U \to \phi(U) \subset \mathbb{R}^n$ and the collection of these charts is called an **atlas**. The composition of one chart with the inverse of another one, defined on the image of non-zero intersection of two neighborhoods under the second chart map, is called a **transition map** which is a homeomorphism between two open subsets of Euclidean space $\mathbb{R}^n$. A manifold is called **smooth manifold** if the transition maps are smooth, i.e. they are all $\mathfrak{C}^\infty$-functions.

Let x be a point in manifold $\mathcal{M}$ with a chart $\phi : U \to \mathbb{R}^n$ where U is an open subset of $\mathcal{M}$ containing x. A **curve** at x is a $\mathfrak{C}^1$ map $c : (-\xi, \xi) \to \mathcal{M}$ from an interval $(-\xi, \xi) \subset \mathbb{R}$ into $\mathcal{M}$ with $c(0) = x$. Let c_1 and c_2 be curves at x such that both $\phi \circ c_1 : (-\xi, \xi) \to \mathbb{R}^n$ and $\phi \circ c_2 : (-\xi, \xi) \to \mathbb{R}^n$ are differentiable at 0. The curves c_1 and c_2 are called tangent at $x \in \mathcal{M}$ if and only if $(\phi \circ c_1)'(0) = (\phi \circ c_2)'(0)$. This defines an equivalence relation on such curves and the *equivalence class* is known as

tangent vector of $\mathcal{M}$ at x. The equivalence class of the curve c is written as $c'(0)$. The set of tangent vectors to $\mathcal{M}$ at x forms a vector space. This is represented by $T_x\mathcal{M}$ and is known as the **tangent space** to $\mathcal{M}$ at x. So we can formally define the tangent space $T_x\mathcal{M}$ as:

$$T_x\mathcal{M} = \{c'(0)|\ c \text{ is a curve at } x \in \mathcal{M}\}$$

The **tangent bundle** of the manifold $\mathcal{M}$, represented by $T\mathcal{M}$, is the disjoint union of the tangent spaces to $\mathcal{M}$ at the points $x \in \mathcal{M}$, i.e,

$$T\mathcal{M} = \bigcup_{x\in\mathcal{M}} T_x\mathcal{M}$$

Thus, an element from $T\mathcal{M}$ is a vector v which is tangent to $\mathcal{M}$ at some point $x \in \mathcal{M}$.

The **cotangent space** at $x \in \mathcal{M}$, represented by $T_x^*\mathcal{M}$, is defined as the dual of the tangent space $T_x\mathcal{M}$, i.e. $T_x^*\mathcal{M} = \{T_x\mathcal{M}\}^*$. Therefore, every element $w \in T_x^*\mathcal{M}$ is a linear functional $w : T_x\mathcal{M} \to \mathbb{R}$. The elements of cotangent space are known as **cotangent vectors** and **cotangent bundle** is defined, similiar to tangent bundle, as the disjoint union of cotangent spaces at the points $x \in \mathcal{M}$

$$T^*\mathcal{M} = \bigcup_{x\in\mathcal{M}} T_x^*\mathcal{M}$$

A.2 Dirac structure

The notion of Dirac structure was first introduced in [3] as a generalized form of symplectic and Poisson structure.

Let $\mathcal{V}$ be an n-dimensional vector space and $\mathcal{W} = \mathcal{V}^*$ be the dual vector space of $\mathcal{V}$ (space of linear functions on $\mathcal{V}$). The product space $\mathcal{V} \times \mathcal{W}$ is considered to be the space of power variables with power defined by

$$P = \langle v|w\rangle \tag{A.1}$$

with $(v, w) \in \mathcal{V} \times \mathcal{W}$ where $\langle \bullet|\bullet\rangle$ denotes the duality product, that is, the linear function $w \in \mathcal{W}$ acts on $v \in \mathcal{V}$.

Now let us consider the following bi-linear form defined over the space of power variables $\mathcal{V} \times \mathcal{W}$

$$\ll (v_1, w_1), (v_2, w_2) \gg = \langle v_1 | w_2 \rangle + \langle v_2 | w_1 \rangle \tag{A.2}$$

with $(v_i, w_i) \in \mathcal{V} \times \mathcal{W}$, $i = 1, 2$. Here $\ll \bullet, \bullet \gg$ is called +pairing operator. This bi-linear form also defines the notion of orthogonality as follows

$$(v_1, w_1) \perp (v_2, w_2) \quad \Rightarrow \quad \ll (v_1, w_1), (v_2, w_2) \gg = 0 \quad \Rightarrow \quad \langle v_1 | w_2 \rangle + \langle v_2 | w_1 \rangle = 0$$

Definition A.2.1 *(**Dirac structure**) Let us consider the space of power variables $\mathcal{F} \times \mathcal{E}$ and the symmetric bi-linear form (A.2). A Dirac structure on $\mathcal{F}$ is a linear subspace $\mathbb{D} \subset \mathcal{F} \times \mathcal{E}$ such that*

$$\mathbb{D} = \mathbb{D}^{\perp} \tag{A.3}$$

The dimension of a Dirac structure $\mathbb{D}$ on an n-dimensional linear space is also equal to n. Now suppose $(f, e) \in \mathbb{D}$, then

$$\ll (f, e), (f, e) \gg = 0 \quad \Rightarrow \quad 2\langle e | f \rangle = 0$$

So we can say $\forall (f, e) \in \mathbb{D}$, $\langle e | f \rangle = 0$ or equivalently, we can say that every Dirac structure $\mathbb{D}$ on $\mathcal{F}$ induces a power-conserving relation between power variables $(f, e) \in \mathcal{F} \times \mathcal{E}$.

A.3 k-forms and wedge products

A **multi-linear form** over a vector space $(\mathcal{V})$ takes vectors from $\mathcal{V}$ and maps them linearly into the field F over which $\mathcal{V}$ is defined. So we can say

$$\omega : \underbrace{\mathcal{V} \times \cdots \times \mathcal{V}}_{k-times} \rightarrow F \tag{A.4}$$

and ω is separately linear for each of its k-arguments. Now a special class of multi-linear forms is **alternating multi-linear forms** which have an additional property of skew-symmetry, that is if we swap any two of its arguments the sign changes.

Therefore for an alternating multi-linear form A

$$A(v_1, \dots, v_i, \dots, v_j, \dots, v_k) = -A(v_1, \dots, v_j, \dots, v_i, \dots, v_k) \tag{A.5}$$

where $v_1, \dots, v_i, \dots, v_j, \dots, v_k \in \mathcal{V}$. Now it can be easily shown that if the arguments are not linearly independent then the alternating multi-linear form (A) will always map to zero. Alternating k-linear forms are simply known as k-forms. Let $\mathcal{T}^k(\mathcal{V})$ represent the set of all possible k-forms over the n-dimensional vector field $\mathcal{V}$. The set $\mathcal{T}^k(\mathcal{V})$ forms a vector space of dimension ${}^nC_k$.

Now let us consider $\mathcal{T}^k(T_x\mathcal{M})$ as the space of all real valued alternating k-linear maps on $T_x\mathcal{M} \times \cdots \times T_x\mathcal{M}$. Using the dual basis dx^i a k-form α can be written as

$$\alpha = \alpha_{i_1,\dots,i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

where the sum is over all i_j satisfying $i_1 < \cdots < i_k$.

If $\alpha \in \mathcal{T}^k(T_x\mathcal{M})$ and $\beta \in \mathcal{T}^l(T_x\mathcal{M})$, then their **tensor product** $\alpha \otimes \beta \in \mathcal{T}^{k+l}(T_x\mathcal{M})$ is defined as

$$(\alpha \otimes \beta)(u_1, \dots, u_k, v_1, \dots, v_l) = \alpha(u_1, \dots, u_k)\beta(v_1, \dots, v_l) \tag{A.6}$$

where $u_1, \dots, u_k, v_1, \dots, v_l \in T_x\mathcal{M}$.

Before we define a wedge product, we need to define another notion - alternation operator. The **alternation operator** ($\mathcal{A} : \mathcal{T}^k(T_x\mathcal{M}) \to \mathcal{T}^k(T_x\mathcal{M})$) is defined as

$$\mathcal{A}(t)(v_1, \dots, v_k) = \frac{1}{k!} \sum_{\pi \in S_p} sgn(\pi) t(v_{\pi(1)}, \dots, v_{\pi(k)}) \tag{A.7}$$

where $t \in \mathcal{T}^k(T_x\mathcal{M})$ and $sgn(\pi)$ is the sign of permutation π

$$sgn(\pi) = \begin{cases} +1 & \text{if } \pi \text{ is even} \\ -1 & \text{if } \pi \text{ is odd} \end{cases}$$

and S_p is the group of all permutations of the set $\{1, 2, \dots, p\}$.

Definition A.3.1 *(**Wedge product**) Let us consider a $(0, k)$ tensor $\alpha \in \mathcal{T}^k(T_x\mathcal{M})$ and $(0, l)$ tensor $\beta \in \mathcal{T}^l(T_x\mathcal{M})$. Then their wedge product $\alpha \wedge \beta \in \mathcal{T}^{k+l}(T_x\mathcal{M})$ is* *defined as*

$$\alpha \wedge \beta = \frac{(k+l)!}{k!l!}\mathcal{A}(\alpha \otimes \beta) \tag{A.8}$$

The wedge product satisfies the following properties

- Bi-linearity:

$$\begin{aligned} \alpha \wedge (a\beta_1 + b\beta_2) &= a(\alpha \wedge \beta_1) + b(\alpha \wedge \beta_2) \\ (a\alpha_1 + b\alpha_2) \wedge \beta &= a(\alpha_1 \wedge \beta) + b(\alpha_2 \wedge \beta) \end{aligned}$$

- Associativity: $\alpha \wedge (\beta \wedge \gamma) = (\alpha \wedge \beta) \wedge \gamma$
- Skew-commutativity: $\alpha \wedge \beta = (-1)^{kl}(\beta \wedge \alpha)$

where $\alpha \in \mathcal{T}^k(T_x\mathcal{M})$ and $\beta, \gamma \in \mathcal{T}^l(T_x\mathcal{M})$.

A.4 Hodge star operator, exterior derivative and Stokes' theorem

The **Hodge star operator** ($\star$) on an n-dimensional inner product space $\mathcal{V}$ is a linear operator on the exterior algebra of $\mathcal{V}$ and it basically transforms a k-form into a $(n-k)$-form [4]. Given an oriented orthonormal basis $e_1, e_2, \ldots, e_n$ on $\mathcal{V}$, we have

$$\star(e_1 \wedge e_2 \wedge \cdots \wedge e_k) = e_{k+1} \wedge e_{k+2} \wedge \cdots \wedge e_n$$

where $0 \leq k \leq n$. The Hodge dual induces an inner product on the space of k-forms. Let ξ and η be two k-forms

$$\xi \wedge \star\eta = < \xi, \eta > \omega \tag{A.9}$$

where ω is the normalized volume form and $< \xi, \eta >$ is the inner product of ξ and η.

Some examples of the operation of the Hodge star operator on $\mathbb{R}^3$ are given below:

- $\star 1 = dx \wedge dy \wedge dz$
- $\star dx = dy \wedge dz$; $\star dy = -dx \wedge dz$

- $\star(dx \wedge dy) = dz$; $\star(dx \wedge dy) = dz$
- $\star(dx \wedge dy \wedge dz) = 1$

The **exterior derivative** $d\alpha$ of a k-form α on a manifold $\mathcal{M}$ is a $(k+1)$-form [12]. The exterior derivative is characterized by the following properties

- If α is a 0-form, that is, $\alpha = f \in \mathfrak{C}^\infty(\mathcal{M})$ then df is the 1-form which is the differential of f.
- $d\alpha$ is linear in α, that is,

$$d(c_1\alpha_1 + c_2\alpha_2) = c_1 d\alpha_1 + c_2 d\alpha_2 \qquad \forall c_1, c_2 \in \mathbb{R}$$

- If α is a k-form and β is an l-form then

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta$$

- $d^2 = 0$, that is, $d(d\alpha) = 0$ for any k-form α.
- d is a local operator, that is, $d\alpha(x)$ only depends on α restricted to any open neighborhood of x; in fact, if U is open in $\mathcal{M}$, then

$$d(\alpha|U) = (d\alpha)|U$$

If α is a k-form and is given in coordinates by

$$\alpha = \alpha_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k} \qquad (\text{sum on } i_1 < \dots < i_k)$$

then the coordinate expression for the exterior derivative $d\alpha$ is given by

$$d\alpha = \frac{\partial \alpha_{i_1 \dots i_k}}{\partial x^j} dx^j \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k} \qquad (\text{sum on all } j \text{ and } i_1 < \dots < i_k)$$

A k-form α is called *closed* if $d\alpha = 0$ and is called *exact* if there exists a $(k-1)$-form β such that $\alpha = d\beta$

Theorem A.4.1 ***(Stokes' theorem)*** *Let us assume that $\mathcal{M}$ is a compact, oriented k-dimensional manifold with boundary $\partial\mathcal{M}$. Let α be a smooth $(k-1)$-form on $\mathcal{M}$.*

Then

$$\int_{\mathcal{M}} d\alpha = \int_{\partial\mathcal{M}} \alpha \tag{A.10}$$

Some special cases of Stokes' theorem are the fundamental theorem of calculus, Green's theorem and the divergence theorem.

A.5 Symplectic manifold and Hamiltonian vector field

A **symplectic form** Ω is a 2-form on a differentiable manifold $\mathcal{M}$ which also satisfies the following properties along with the properties of a 2-form

- $\mathbf{d}\Omega = 0$ (*i.e.* exterior derivative of Ω is zero)
- If $\Omega_x(v_1, v_2) = 0 \quad \forall v_1 \in T_x\mathcal{M}$ and a fixed $v_2 \in T_x\mathcal{M}$ then $v_2 = 0$ (*i.e.* Ω is non-degenerative).

A differentiable manifold $\mathcal{M}$ with a symplectic form Ω is called a **symplectic manifold** and is denoted by $(\mathcal{M}, \Omega)$.

Now let us consider a symplectic manifold $\mathcal{M}$ with the symplectic form Ω. Let f be a smooth function over $\mathcal{M}$, that is $f \in \mathfrak{C}^\infty(\mathcal{M})$. Let X_f be the unique vector field on $\mathcal{M}$ which satisfies

$$\Omega_x(X_f(x), v) = df(x).v \quad \forall v \in T_x\mathcal{M} \tag{A.11}$$

Then X_f is called the **Hamiltonian vector field** of f [12].

A.6 Poisson manifold

A **Poisson bracket** on a manifold $\mathcal{M}$ is a bilinear operation $\{\cdot,\cdot\}$ on $\mathcal{F}(\mathcal{M}) = \mathfrak{C}^\infty(\mathcal{M})$ such that

- $\{\cdot,\cdot\} : \mathcal{F}(\mathcal{M}) \times \mathcal{F}(\mathcal{M}) \to \mathcal{F}(\mathcal{M})$
- $(\mathcal{F}(\mathcal{M}), \{\cdot,\cdot\})$ is a Lie algebra; and

- $\{\cdot,\cdot\}$ is a derivation in each factor, that is

$$\{FG, H\} = \{F, H\}G + F\{G, H\} \quad \forall F, G, H \in \mathcal{F}(\mathcal{M})$$

A Poisson bracket is also sometimes known as a **Poisson structure**.

A manifold $\mathcal{M}$ endowed with a Poisson bracket on $\mathcal{F}(\mathcal{M})$ is called a **Poisson manifold** and is denoted by $(\mathcal{M}, \{\cdot,\cdot\})$. It can be noted that any manifold $\mathcal{M}$ has the *trivial Poisson structure* defined by $\{F, G\} = 0$ for all $F, G \in \mathcal{F}(\mathcal{M})$.

Another thing to be noted is that *any symplectic manifold* $(\mathcal{M}, \Omega)$ *is a Poisson manifold*. The Poisson bracket is defined by the symplectic form as

$$\{F, G\}(x) = \Omega_x(X_F(x), X_G(x))$$

A.7 Variational derivative

Consider a density function $G : \Omega^k(\mathcal{M}) \times \mathcal{M} \to \Omega^n(\mathcal{M})$ where $\Omega^k(\mathcal{M})$ is the space of all differential k-forms on a n-dimensional manifold $\mathcal{M}$ and $k \in \{1, 2, \dots, n\}$. Now we introduce the functional

$$\mathcal{G} \triangleq \int_{\mathcal{M}} G \in \mathbb{R}$$

Then the uniquely defined differential form $\frac{\delta \mathcal{G}}{\delta \alpha} \in \Omega^{n-k}(\mathcal{M})$ which satisfies

$$\mathcal{G}(\alpha + \epsilon\Delta\alpha) = \int_{\mathcal{M}} G(\alpha + \epsilon\Delta\alpha) = \int_{\mathcal{M}} G(\alpha) + \epsilon \int_{\mathcal{M}} \left[\frac{\delta \mathcal{G}}{\delta \alpha} \wedge \Delta\alpha\right] + O(\epsilon^2)$$

for all $\Delta\alpha \in \Omega^k(\mathcal{M})$ and $\epsilon \in \mathbb{R}$, is called the **variational derivative** of $\mathcal{G}$ with respect to $\alpha \in \Omega^k(\mathcal{M})$. This is also represented by $\delta_\alpha \mathcal{G}$.

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