Synchronization in Neuronal Oscillator Networks

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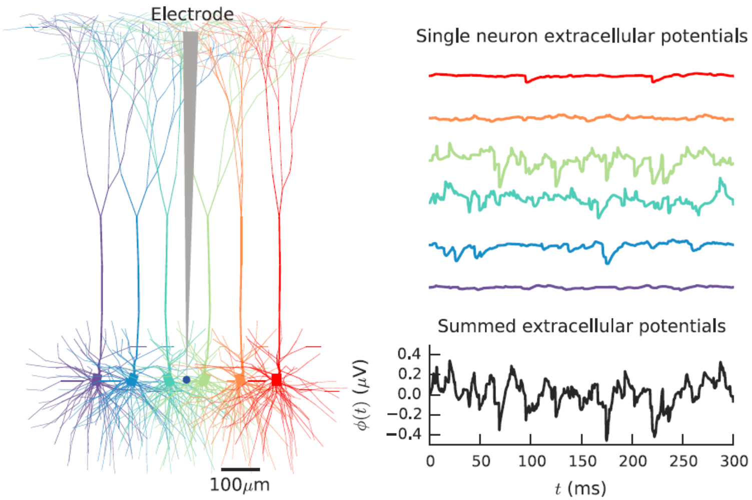
Motivation

- ❑ Synchronized activity is crucial for brain function:
  - ❑ Basal ganglia
  - ❑ Local Field Potential
  - ❑ fMRI/functional connectivity

- ❑ Knowledge about conditions for synchronization can lead to a better understanding of:
  - ❑ Deep Brain Stimulation
  - ❑ Transcranial Stimulation
  - ❑ System Identification
  - ❑ Testable predications
  - ❑ Measurable efficacy metrics for disease treatment



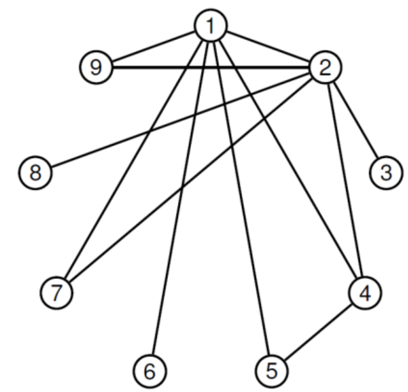
Linden et al. (2014), *LFPy: a tool for biophysical simulation of extracellular potentials generated by detailed model neurons,* Frontiers in Neuroinformatics.

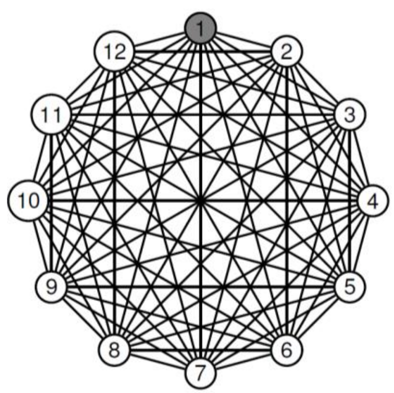
MIT Tech. Review (March 2016), Halo Neuroscience

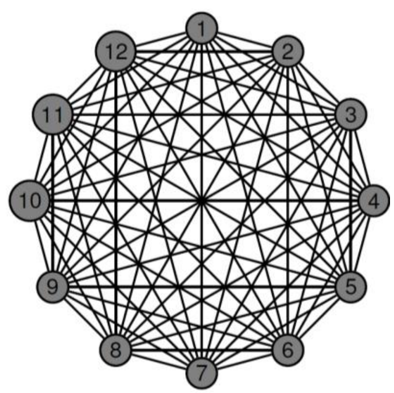
Synchronization in Networks

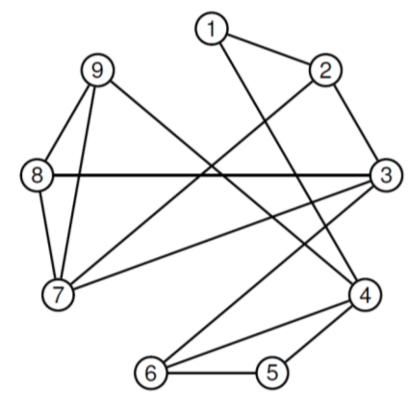
- ❑ Asymptotically stable synchronization in a network of homogeneous *semi-passive* neuronal oscillators is guaranteed with sufficient coupling.[1]

- ❑ Changes in the stability of synchronization/ consensus manifold result from:
  - ❑ Graph Structure
  - ❑ Coupling
  - ❑ External Inputs
  - ❑ Time Delay
  - ❑ Oscillator properties









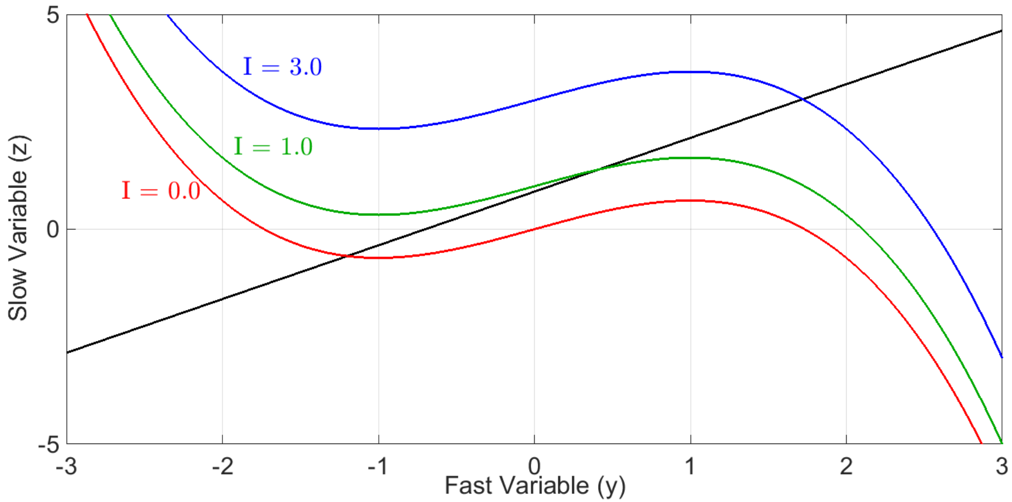
[1] E. Steur, I. Tyukin, and H. Nijmeijer (2009), *Semi-passivity and synchronization of diffusively coupled neuronal oscillators*, Physica D: Nonlinear Phenomena, 238(21):2119 - 2128.

Neuronal Oscillator: Fitzhugh-Nagumo (FN)

- Second order dynamics for membrane potential
  - Fast Dynamics: $\dot{y} = y - \frac{y^3}{3} - z + I$
  - Slow Dynamics: $\dot{z} = \epsilon(y - bz + a)$

I → External Input

ϵ → Time scale separation



The dynamics of FN Oscillator model is strictly semi-passive.

- Outside a ball around the origin, a strictly semi-passive system behaves as a strictly passive system.

Network of FN Oscillators

Dynamics of a Single Neuron in the Network

- $\dot{y}_i = y_i - \frac{y_i^3}{3} - z_i + I_i + u_i$
- $\dot{z}_i = \epsilon(y_i - bz_i + a)$

Social Influence (u_i)

Electrical gap junction coupling:

- $u_i = \sum_{j=1}^{n} \gamma_{ij}(y_j - y_i), \;\; \gamma_{ij} \geq 0$

$$\Rightarrow \quad \mathbf{u} = -\Gamma \mathbf{y} \qquad \Gamma = \begin{bmatrix} \sum \gamma_{1j} & -\gamma_{12} & \cdots & -\gamma_{1n} \\ -\gamma_{21} & \sum \gamma_{2j} & \cdots & -\gamma_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ -\gamma_{n1} & -\gamma_{n2} & \cdots & \sum \gamma_{nj} \end{bmatrix}$$

The closed-loop system has ultimately bounded solutions.[2]

- In finite time, solutions of the closed-loop system enter a compact set that is invariant under the system dynamics.

[2] A. Pogromsky, T. Glad, and H. Nijmeijer (1999), *On diffusion driven oscillations in coupled dynamical systems*, International Journal of Bifurcation and Chaos, 9(4):629 - 644, 1999E.

A Sufficient Condition for Synchronization

- Lyapunov Function:

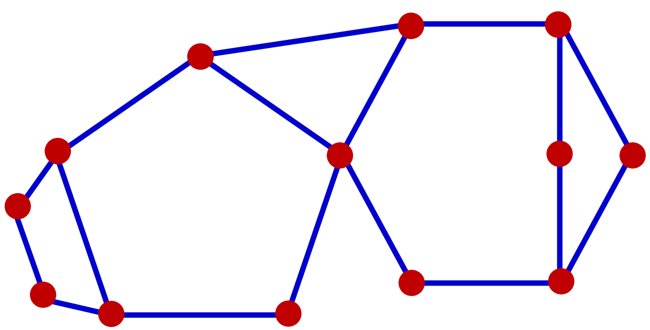
$$V = \frac{1}{2}\sum_{j=2}^{n}\left[(y_j - y_1)^2 + (z_j - z_1)^2\right]$$

- Lyapunov Theorem exploits bounds (arising out of semi-passivity) on solution, i.e.

$$|y_i| < \beta_y,\ \forall i \in \{1, 2, \cdots, n\}$$

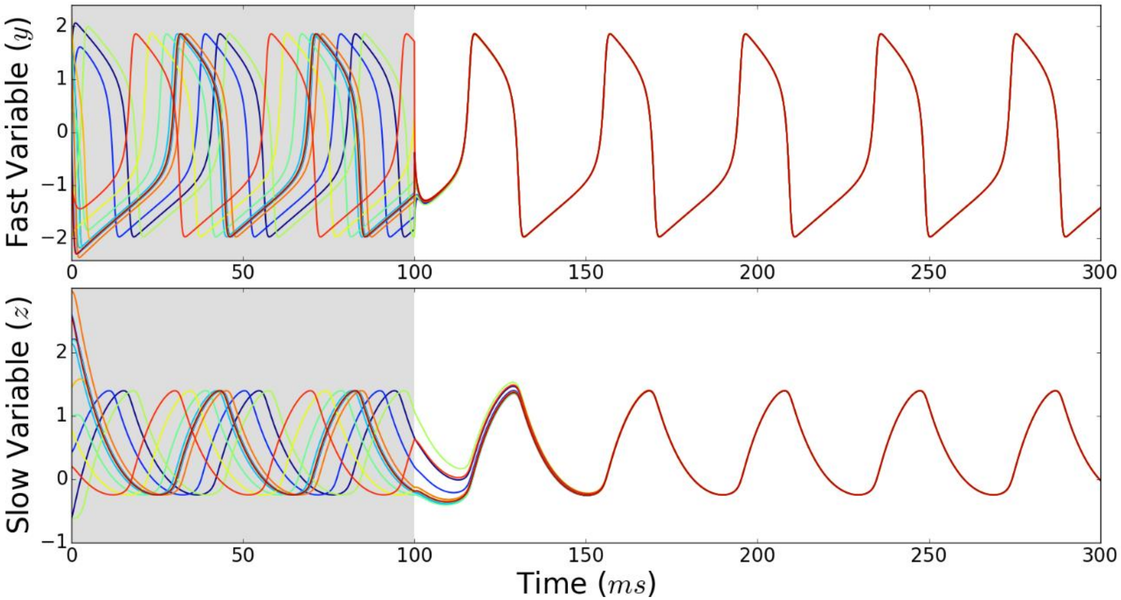
- Lower bound on second-smallest eigenvalue of the graph Laplacian.

$$\lambda_2(\Gamma) > 1 + \frac{1}{3}\beta_y^2 + \frac{1}{4b}\left(\epsilon + \frac{1}{\epsilon} - 2\right) \triangleq \lambda_s^*$$



$a = 0.7, b = 0.8, \epsilon = \frac{1}{12.5}, I = 0.5$

$$\lambda_2(\Gamma) = 2.29 < 5.45 = \lambda_s^*$$



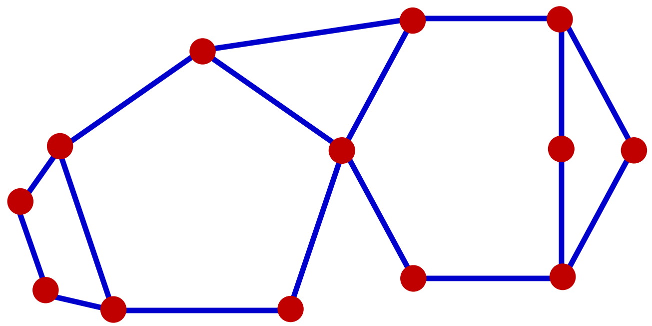
A Tighter Bound

- Non-smooth Lyapunov Function:

$$V = \max_{i,j\in\{1,\cdots,n\}} |y_i - y_j| + \max_{i,j\in\{1,\cdots,n\}} |z_i - z_j|$$

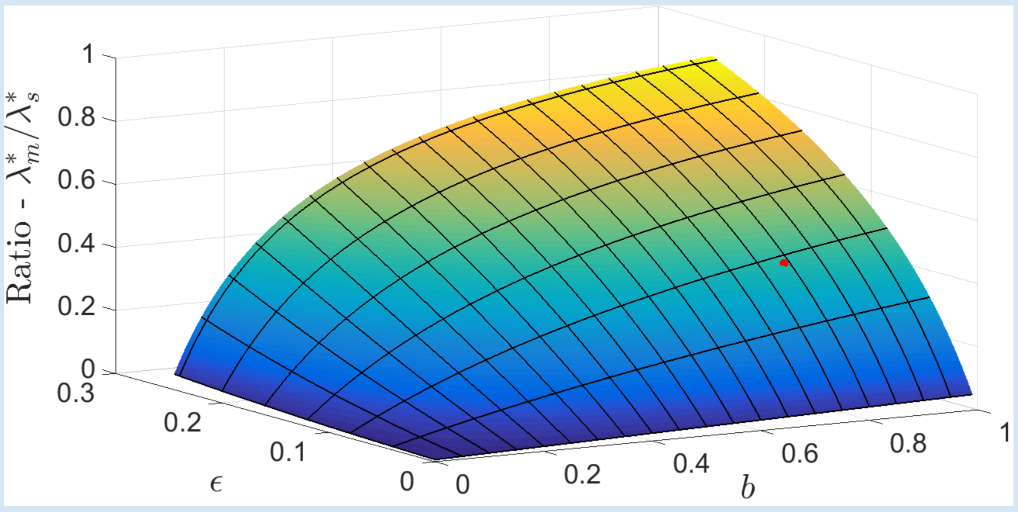
- Lower bound on second-smallest eigenvalue of the graph Laplacian.

$$\lambda_2(\Gamma) > 1 + \frac{1}{3}\beta_y^2 + \epsilon \triangleq \lambda_m^*$$



$$\lambda_2(\Gamma) = 2.29 > 2.22 = \lambda_m^*$$

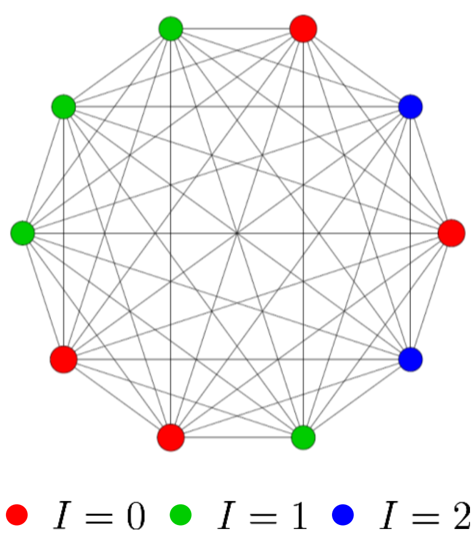
Bound Comparison

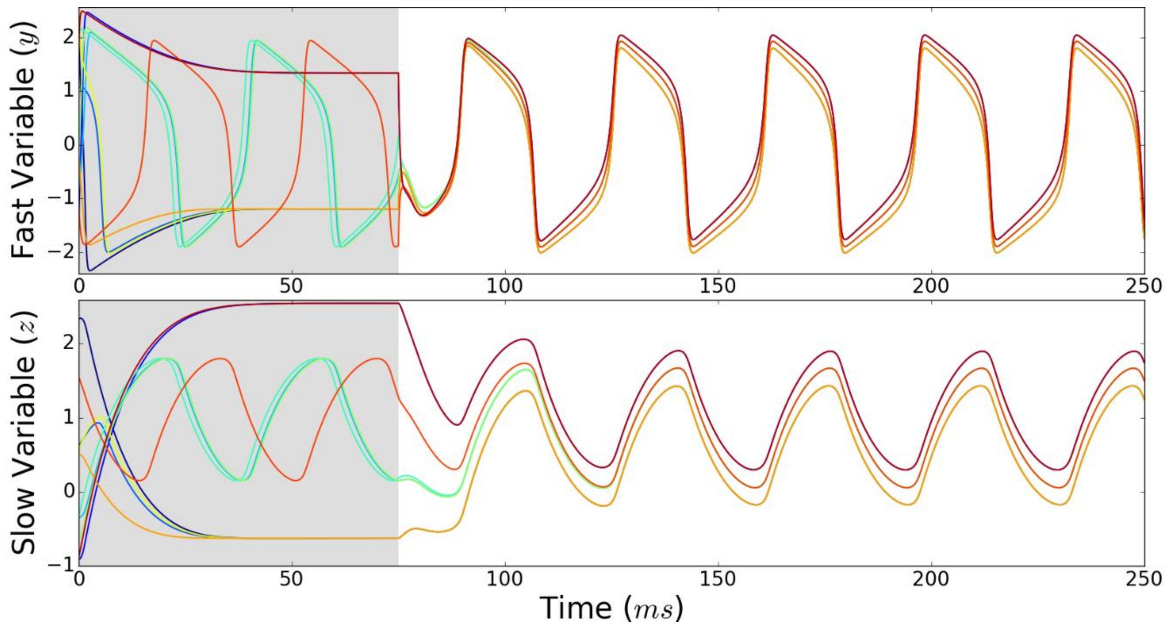


Input Heterogeneity in a Complete Graph

- Synchronization is only possible when the sum of external input and social influence are same across the individuals.
- Input heterogeneity gives rise to multiple clusters in the network.
  - Clusters are determined by input structure.
- A sufficient condition for synchronization of individual clusters in a complete graph, with $\gamma_{ij} = \gamma > 0, i \neq j$ and $\gamma_{ii} = 0$.

$$\gamma > \frac{3 + \beta_y^2 + 3\epsilon}{3n}$$





Reduction in a Complete Graph

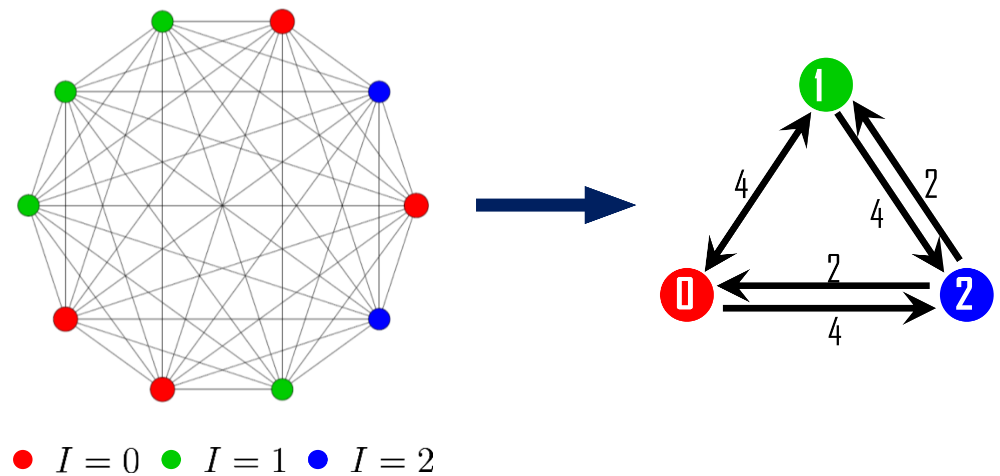
- Change of Coordinates:
  - Average:

$$\xi_1 = \frac{1}{k}\sum_{j=1}^{k} y_j \qquad \zeta_1 = \frac{1}{k}\sum_{j=1}^{k} z_j$$

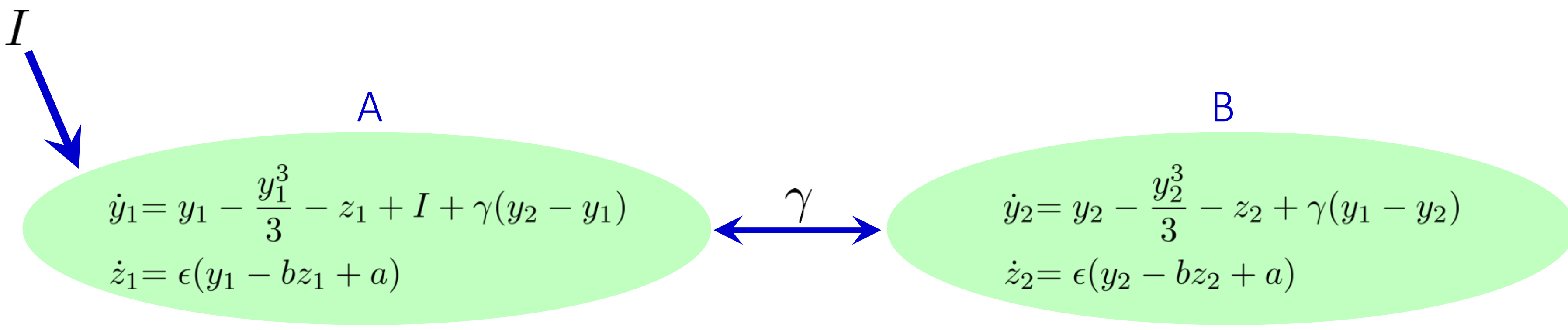
  - Difference from Average:

$$\xi_i = y_i - \frac{1}{k}\sum_{j=1}^{k} y_j \qquad \zeta_i = z_i - \frac{1}{k}\sum_{j=1}^{k} z_j \qquad i \in \{2, 3, \cdots, k\}$$

- When the coupling is strong enough for synchronization of individual oscillators, the dynamics of the average (of membrane potential and recovery variable) becomes identical to the dynamics of a single oscillator.



Entrainment in a Two Oscillator Network



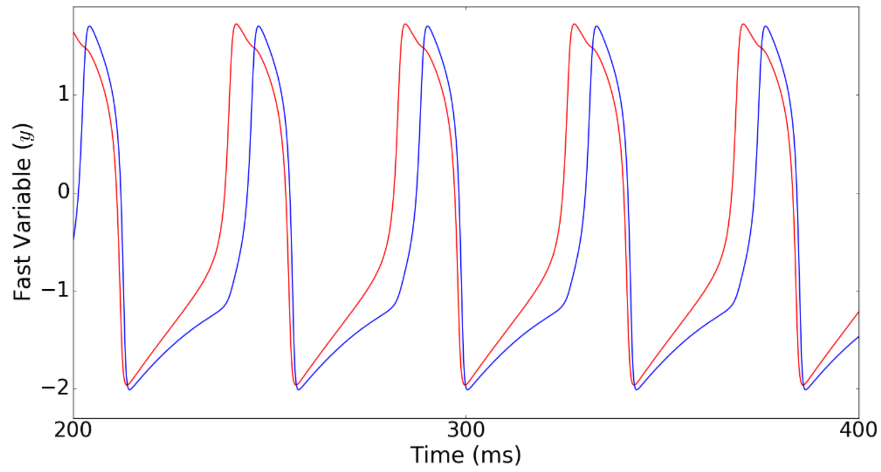
The following regimes exist in this framework:

- When $I < I_0$, both A and B are quiescent.
- When $I_0 < I < I_1$
  - and $\gamma < \gamma_{f0}$, A is firing and B is quiescent.
  - and $\gamma_{f0} < \gamma < \gamma_{f1}$, both A and B are firing.
  - and $\gamma > \gamma_{f1}$, both A and B become quiescent again.
- When $I > I_1$
  - and $\gamma < \gamma_{s0}$, A is saturated and B is quiescent.
  - and $\gamma > \gamma_{s0}$, both A and B are firing.

Conclusion

- Sufficient condition for synchronization in networks of homogeneous FitzHugh-Nagumo oscillators.

- Emergence of cluster synchronization due to *input heterogeneity* in a complete graph.



Thank You!